Algebraic Geometry 1 (5) (2014) 558–572
doi:10.14231/AG-2014-024

Very free curves on Fano complete intersections

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Abstract

In this paper, we show that general Fano complete intersections over an algebraically closed field of arbitrary characteristic are separably rationally connected. Our construction of rational curves leads to a more interesting generalization that general log Fano complete intersections with smooth tame boundary divisors admit very free $\mathbb{A}^1$-curves.

1. Introduction

1.1 Background and main results

The existence of rational curves in higher dimensional varieties shapes their geometry to a large extent. The following definitions describe the existence of many rational curves.

Definition 1.1 [Kol96, Chapter IV, Section 3]. Let X be a variety defined over an arbitrary field k.

(i) A variety X is *rationally connected* (RC) if there exist a family of irreducible proper rational curves $g : U \to Y$ and a cycle morphism $u : U \to X$ such that the morphism $u^{(2)} : U \times_Y U \to X \times X$ is dominant.

(ii) A variety X is *rationally chain connected* (RCC) if there exist a family of chains of rational curves $g : V \to Y$ and a cycle morphism $u : V \to X$ such that the morphism $u^{(2)} : V \times_Y V \to X \times X$ is dominant.

(iii) A variety X is *separably rationally connected* (SRC) if over the algebraic closure $\overline{k}$, there exists a proper rational curve $f : \mathbb{P}^1 \to X$ such that X is smooth along the image and f^*T_X is ample. Such rational curves are called *very free* curves.

We refer to Kollár's book [Kol96] for background information. The third definition is stronger than [Kol96, IV 3.2.3], but they coincide for smooth varieties. It is known that SRC implies RC and RC implies RCC. All these notions of rational connectedness are equivalent for smooth varieties in characteristic zero. However, in positive characteristic it is known that RC is strictly weaker than SRC.

The fundamental results of Campana [Cam92] and Kollár, Miyaoka and Mori [KMM92] show that Fano varieties, that is, smooth varieties with ample anticanonical bundles, are rationally chain connected. In particular, Fano varieties are SRC in characteristic zero.

Received 5 February 2014, accepted in final form 2 June 2014.
2010 Mathematics Subject Classification 14M22 (primary), 14E99 (secondary)
Keywords: Fano complete intersection, rationally connectedness, logarithmic geometry.

Chen is supported by the Simons foundation.

It has been pointed out by Kollár that separable rational connectedness is the right notion for rational connectedness in arbitrary characteristic. They have both nice geometric and nice arithmetic applications.

Geometrically, when the base field is algebraically closed, we have the following results:

(1) Graber, Harris and Starr [GHS03] prove the famous theorem asserting that in characteristic zero, a proper family of varieties over an algebraic curve whose general fiber is smooth RC admits a section.

(2) de Jong and Starr [dJS03] generalize the result of [GHS03] by showing the existence of sections in positive characteristic when general fibers are smooth SRC.

(3) The weak approximation for families of separably rationally connected varieties was studied in [HT06], [HT08a].

(4) Tian and Zong [TZ12] show that the Chow group of 1-cycles on a smooth proper SRC variety is generated by rational curves.

Arithmetically, we have the following results:

(1) When the base field is a local field, Kollár [Kol99] shows that a smooth proper SRC variety admits a very free curve through any rational point.

(2) When the base field is a finite field of cardinality $q \leqslant \infty$, Kollár and Szabó [KS03] show that there is a function $\Phi : \mathbb{N}^3 \to N$ such that for a smooth projective SRC variety $X \subset \mathbb{P}^N$, given any zero-dimensional subscheme $S \subset X$, there exists a smooth rational curve on X containing S whenever $q > \Phi(\deg X, \dim X, \deg S)$.

(3) When the base field is a large field, Hu [Hu10] proves interesting results on the weak approximation conjecture for SRC varieties at places of good reduction.

Despite the nice behavior of SRC varieties, for many important varieties which are known to be RC in characteristic zero it is difficult to verify the SRC condition in positive characteristic. The following question is the major motivation for the present paper.

Question 1.2 (Kollár). In arbitrary characteristic, is every smooth Fano variety separably rationally connected?

Notation 1.3. Since Question 1.2 can be checked over the algebraic closure, for the rest of this paper, we work with algebraic varieties over an algebraically closed field $\mathbf{k}$ of arbitrary characteristic.

The first test case is Fano complete intersections in projective space. The difficulty is to prove separable rational connectedness in low characteristic [KS03, Conjecture 14].

The answer to the question is known for general Fano hypersurfaces by [Zhu11], where very free curves are constructed explicitly over degenerate Fano varieties. In this paper, we provide an answer in the complete intersection case.

Theorem 1.4. *In arbitrary characteristic, a general Fano complete intersection in $\mathbb{P}^n$ is separably rationally connected.*

Remark 1.5. During the preparation of this paper, the authors learned another interesting proof of Theorem 1.4 by Zhiyu Tian [Tia13] using a different method.

Our theorem eliminates the SRC condition in [TZ12, Theorem 1.7].

COROLLARY 1.6. *Let X be a general complete intersection of type $(d_1, \cdots, d_l)$ in $\mathbb{P}^n$ such that $d_1 + \cdots + d_l \leqslant n - 1$. Then the Chow group of 1-cycles on X is generated by lines.*

Surprisingly, our construction of very free curves leads to a much stronger generalization of Theorem 1.4, as follows.

THEOREM 1.7. *In arbitrary characteristic, a general log Fano complete intersection with a general tame boundary in $\mathbb{P}^n$ is separably $\mathbb{A}^1$-connected.*

We refer to Sections 1.2 and 1.3 for more details of the definitions and the proof of the results given above.

Note that the $\mathbb{A}^1$-connectedness implies the rational connectedness of the underlying variety. However, the other direction fails in general. In fact, a log Fano log variety needs not be $\mathbb{A}^1$-connected! For example, the log variety associated with $\mathbb{P}^2$ with boundary given by two distinct lines fails to be $\mathbb{A}^1$-connected. On the other hand, producing an $\mathbb{A}^1$-curve is much more difficult than producing rational curves, due to the constraints on the boundary marking. Thus, Theorem 1.7 turns out to be a more interesting result.

Question 1.8. Can we drop the tame condition on the boundary in Theorem 1.7?

Our construction of very free log maps provides a much more general result in characteristic zero, as follows.

THEOREM 1.9. *Assume* char $\mathbf{k} = 0$. *Let (X, D) be a log Fano smooth pair; that is, the boundary divisor D is smooth and irreducible, and $-(K_X + D)$ is ample. Then X is separably $\mathbb{A}^1$-connected if and only if it is separably $\mathbb{A}^1$-uniruled.*

COROLLARY 1.10. *Let (X, D) be a log Fano smooth pair as in Theorem 1.9, and let* char $\mathbf{k} = 0$. *If the divisor class of the normal bundle of D is numerically equivalent to a nontrivial effective divisor, then (X, D) is $\mathbb{A}^1$-connected.*

The proofs of Theorem 1.9 and Corollary 1.10 will be given at the end of Section 4.2.

The condition in this corollary was first stated in the work of Hassett–Tschinkel [HT08b]. The $\mathbb{A}^1$-uniruledness condition seems to be a more natural setting—it includes, for example, $(\mathbb{P}^1, \{\infty\})$ and Hirzebruch surfaces with the negative curve as the boundary, which are $\mathbb{A}^1$-uniruled, but for which the divisor class of the normal bundle of the boundary divisor is either trivial or noneffective.

Keel and McKernan [KM99] prove that any log Fano pair over the complex numbers is either $\mathbb{A}^1$-uniruled or uniruled. It is natural to ask the following question.

Question 1.11. Let (X, D) be a log smooth log Fano variety with D irreducible. Is the pair (X, D) always $\mathbb{A}^1$-uniruled?

Remark 1.12. It should be emphasized that our proof of Theorems 1.4 and 1.7 is constructive, which allows one to write down the exact degree of the very free curves in each case. We leave the details of this to interested readers.

On the other hand, it seems to us that $\mathbb{A}^1$-connectedness itself is a very useful concept for the study of quasi-projective varieties. The results of Theorems 1.7 and 1.9 provide many interesting and concrete examples for $\mathbb{A}^1$-connectedness. In our papers [CZ14a] and [CZ14b], we will study the properties of $\mathbb{A}^1$-connectedness for general log smooth varieties; following the work of [HT08b], we will consider an application to the Zariski density of integral points of curves over function fields.

We next summarize the ideas used in the proofs of Theorems 1.4 and 1.7.

1.2 Log Fano complete intersections

Let (X, D) be a smooth pair consisting of a smooth variety X and a smooth divisor $D \subset X$. The *divisorial log structure* on X is defined as

$$\mathcal{M}_X := \{s \in \mathcal{O}_X \mid s|_{X \setminus D} \in \mathcal{O}^*\}. \tag{1.2.1}$$

Let $X^\dagger = (X, \mathcal{M}_X)$ be the log scheme defined by the smooth pair (X, D). In this case, $X^\dagger$ is log smooth. We refer to [Kat89] for the basic terminology of logarithmic geometry. When there is no danger of confusion, we may use (X, D) for the log scheme $X^\dagger$ to specify the boundary D.

Consider the projective space $\mathbb{P}^n$ with homogeneous coordinates $\bar{x} = [x_0 : \cdots : x_n]$. Throughout this paper, we fix a sequence of nonnegative integers

$$d_1, \cdots, d_l, d_b \tag{1.2.2}$$

such that $d_b + \sum_{i=1}^{l} d_i \leqslant n$. We further require that d_i be positive for any $i = 1, \cdots, l$. Note that we allow $d_b = 0$. Choose a collection of general homogeneous polynomials in $\bar{x}$:

$$F_1, F_2, \cdots, F_l, G$$

with degrees $\deg G = d_b$ and $\deg F_i = d_i$ for all i.

Let $X \subset \mathbb{P}^n$ be the subscheme defined by F_i for $i = 1, \cdots, l$, and let $D \subset X$ be the locus cut out by G. Since G and F_i are general, we may assume that (X, D) is a smooth pair. We call the corresponding log scheme $X^\dagger$ a *log* $(d_1, \cdots, d_l; d_b)$*-complete intersection*. We say that $X^\dagger$ has a *tame boundary* if $\operatorname{char} \mathbf{k} \nmid d_b$. When $d_b = 0$, the variety X is a Fano $(d_1, \cdots, d_l)$-complete intersection in the usual sense.

1.3 Separable $\mathbb{A}^1$-connectedness

Let $\mathcal{X}^\dagger \to B^\dagger$ be a morphism of log schemes. A *stable log map* over a log scheme $S^\dagger$ is a commutative diagram

$$\begin{tikzcd} C^\dagger \arrow[r, "f"] \arrow[d] & \mathcal{X}^\dagger \arrow[d] \\ S^\dagger \arrow[r] & B^\dagger \end{tikzcd} \tag{1.3.1}$$

such that $C^\dagger \to S^\dagger$ is a family of log curves over $S^\dagger$ as defined in [Kat96, Ols07] and the underlying map $\underline{f}$ is a family of usual stable maps to the underlying family of targets $\mathcal{X}/B$.

The theory of stable log maps has been developed by Gross–Siebert [GS13] and independently by Abramovich–Chen [Che10, AC11]. The most important result about log maps we will need in this paper is that they form an algebraic stack. Both [GS13] and [Che10] assume to work over a field of characteristic zero for the purpose of Gromov–Witten theory, but the proof of algebraicity works in general. Olsson's log cotangent complex [Ols05] provides a well-behaved deformation theory for studying stable log maps when $\mathcal{X}^\dagger \to B^\dagger$ is log smooth.

For a log smooth scheme $X^\dagger$ over $\operatorname{Spec} \mathbf{k}$, we write $\Omega_{X^\dagger}$ and $T_{X^\dagger} = \Omega_{X^\dagger}^\vee$ for the log cotangent and tangent bundles, respectively. Generalizing Definition 1.1(3), we introduce the terminology which is crucial to our construction.

Definition 1.13. A log scheme $X^\dagger$ given by a divisorial log smooth pair (X, D) is called *separably*

$\mathbb{A}^1$-*connected* or *separably* $\mathbb{A}^1$-*uniruled* if there is a single-marked, genus zero log map

$$f : C^\dagger/S^\dagger \to X^\dagger$$

with $C \cong \mathbb{P}^1$ and S a geometric point, such that $f^*T_{X^\dagger}$ is ample or semipositive, respectively, and the tangency at the marking is nontrivial. We call such log stable map a *very free* $\mathbb{A}^1$-*curve* or *free* $\mathbb{A}^1$-*curve*, respectively.

Remark 1.14. (1) Since the tangency at the marking is nontrivial, the image of the marked point has to lie on the boundary D. We refer to [ACGM10] for the canonical evaluation spaces of the markings.

(2) The definition of log maps allows the image of components of the source curve to lie on the boundary divisor. But when $f^*T_{X^\dagger}$ is semipositive, a general deformation yields a map with smooth source curve whose image meets the boundary divisor only at the markings; see Lemma 3.8. Thus, the definition of $\mathbb{A}^1$-uniruledness given above is compatible with the definition in [KM99].

1.4 Proof of Theorems 1.4 and 1.7

As in [Zhu11], the approach we will use here is by taking degenerations. However, this time we are able to chase the deformation theory with the help of logarithmic geometry. We summarize the steps in the proof, and refer to later sections for the technical details.

First, consider a general Fano $(d_1, \cdots, d_l)$-complete intersection X. We take a general simple degeneration of X as in Section 3.1, and obtain a singular fiber by gluing a general log Fano $(d_1, \cdots, d_l - 1; 1)$-complete intersection (X_1, D) and a general log Fano $(d_1, \cdots, 1; d_l - 1)$-complete intersection (X_2, D) along the boundary divisor D.

Observation 1.15 (See Proposition 3.9). The general fiber is SRC if

(i) the complete intersection (X_1, D) is separably $\mathbb{A}^1$-connected;

(ii) the complete intersection (X_2, D) is separably $\mathbb{A}^1$-uniruled.

Second, note that D is a general Fano $(d_1, \cdots, d_{l-1}, d_l - 1, 1)$-complete intersection in dimension one less. We reduce both (i) and (ii) in Observation 1.15 to the SRC property of the boundary D, as follows.

Observation 1.16 (See Lemma 4.2 and Proposition 4.3).

(i) The complete intersection (X_1, D) is separably $\mathbb{A}^1$-connected if D is ample and SRC.

(ii) The complete intersection (X_2, D) is separably $\mathbb{A}^1$-uniruled if D is ample and separably uniruled.

Finally, the inductive process ends when D is a projective space, which is of course separably rationally connected.

2. Free $\mathbb{A}^1$-lines on log Fano complete intersections

The following is a variation of [Ang12, Theorem 4.3], which allows us to construct free $\mathbb{A}^1$-lines explicitly on log Fano complete intersections.

Lemma 2.1. *Let X be a smooth $(d_1, \cdots, d_l)$-complete intersection in $\mathbb{P}^n$ defined by*

$$\{F_1 = \cdots = F_l = 0\}$$

and let $D = \cup_{j=1}^{k} D_j$ be a simple normal crossing divisor on X with each irreducible component D_j defined by $\{G_j = 0\}$, $1 \leqslant j \leqslant k$. Let $X^\dagger$ be the log variety associated with the pair (X, D), and write $d'_j = \deg G_j$. Then the log tangent bundle $T_{X^\dagger}$ is the middle cohomology of the following complex:

$$\mathcal{O}_X \xrightarrow{\ A\ } \mathcal{O}_X(1)^{\oplus(n+1)} \oplus \mathcal{O}_X^{\oplus l} \xrightarrow{\ B\ } \sum_{i=1}^{l} \mathcal{O}_X(d_i) \oplus \sum_{j=1}^{k} \mathcal{O}_X(d_j)\,,$$

with the arrows defined by

$$A = (x_0, \cdots, x_n, d'_1, \cdots, d'_k)^T$$

and

$$B = \begin{pmatrix} \operatorname{Jac} \vec{F} & 0 \\ \operatorname{Jac} \vec{G} & \operatorname{diag}(\vec{G}) \end{pmatrix},$$

where $\vec{F} = (F_1, \cdots, F_l)$, $\vec{G} = (G_1, \cdots, G_k)$, $\operatorname{Jac} \vec{F}$ and $\operatorname{Jac} \vec{G}$ are the corresponding Jacobian matrices and $\operatorname{diag}(\vec{G})$ denotes the diagonal matrix.

Furthermore, when $\operatorname{char} \mathbf{k} \nmid d'_j$ for all j, the log tangent bundle $T_{X^\dagger}$ is given by the kernel of the morphism

$$\mathcal{O}_X(1)^{\oplus(n+1)} \oplus \mathcal{O}_X^{\oplus(k-1)} \xrightarrow{\ B'\ } \sum_{i=1}^{l} \mathcal{O}_X(d_i) \oplus \sum_{j=1}^{k} \mathcal{O}_X(d_j)\,,$$

where

$$B' = \begin{pmatrix} \operatorname{Jac} \vec{F} & 0 \\ \operatorname{Jac} \vec{G} & \operatorname{diag}(G_1, \cdots, G_{k-1}) \end{pmatrix}.$$

Proof. In the case $\vec{F} = 0$, the result is proved in [Ang12, Theorem 4.2]. In the case $\vec{G} = 0$ and $\vec{F}$ nontrivial, the tangent bundle T_X is given by the middle cohomology of the following complex:

$$\mathcal{O}_X \xrightarrow{\ A\ } \mathcal{O}_X(1)^{\oplus(n+1)} \xrightarrow{\operatorname{Jac} \vec{F}} \sum_{i=1}^{l} \mathcal{O}_X(d_i)\,.$$

The statement now follows by combining this sequence with [Ang12, Theorem 4.2]. □

Proposition 2.2. *Let (X, D) be a general log Fano $(d_1, \cdots, d_l; d_b)$-complete intersection in $\mathbb{P}^n$ with $e = \sum_{i=1}^{l} d_i + d_b \leqslant n$. If $\operatorname{char} \mathbf{k} \nmid d_b$, then the pair (X, D) is separably log-uniruled by lines. Furthermore, the restriction of the log tangent bundle to a general log free line has splitting type*

$$\mathcal{O}(1)^{\oplus(n+1-e)} \oplus \mathcal{O}^{\oplus(e-l-1)}\,.$$

Proof. By log deformation theory, it suffices to produce a pair (X, D), log-smooth along a line, such that the restriction of the log tangent bundle is semipositive.

Let L be the line defined by $\{x_2 = \cdots = x_n = 0\}$. For simplicity, we introduce $m_j = \sum_{i=1}^{j} d_i$ and set $m_0 = 0$ and $m_{l+1} = e$. Choose the following homogeneous polynomials:

$$\begin{array}{rcl} F_i &=& x_{m_{i-1}+2} \cdot x_0^{d_i-1} + x_{m_{i-1}+3} \cdot x_1 \cdot x_0^{d_i-2} + \cdots + x_{m_i+1} \cdot x_1^{d_i-1} \\ G &=& x_1^{d_b} + x_{m_l+2} \cdot x_1^{d_b-2} \cdot x_0 + \cdots + x_{m_{l+1}} \cdot x_0^{d_b-1} \end{array} \tag{2.0.1}$$

for $i \in \{1, \cdots, c\}$. Note that when $d_b = 1$, we get

$$G = x_1 \tag{2.0.2}$$

and all F_i remain the same. We then check that

(i) the line L lies in the smooth locus of X;

(ii) the line L intersects D only at the point $[x_0 : x_1] = [1 : 0]$, which is a smooth point of D.

By Lemma 2.1, we have a short exact sequence of sheaves over L after twisting down by $\mathcal{O}(-1)$:

$$0 \longrightarrow T_{X^\dagger}|_L(-1) \longrightarrow \mathcal{O}^{\oplus(n+1)} \xrightarrow{B'} \sum_{i=1}^{c} \mathcal{O}(d_i - 1) \oplus \mathcal{O}(d_b - 1) \longrightarrow 0\,.$$

Note that $T_{X^\dagger}|_L$ is semipositive if and only if $H^1(L, T_{X^\dagger}|_L(-1)) = 0$. By the long exact cohomology sequence, it suffices to show that

$$H^0(\mathcal{O}^{\oplus(n+1)}) \xrightarrow{B'} H^0(\sum_{i=1}^{c} \mathcal{O}(d_i - 1) \oplus \mathcal{O}(d-1))$$

is surjective. This follows from the assumption char $\mathbf{k} \nmid d_b$ and the choice of polynomials F_i and G. Finally, since $T_{X^\dagger}|_L$ is a subsheaf of $\mathcal{O}(1)^{\oplus n+1}$, the splitting type of the log tangent bundle on the line is as desired. This finishes the proof. $\square$

3. Reduction to log Fano varieties via degeneration

3.1 Simple degeneration

Consider a log smooth morphism of fine and saturated log schemes $\pi : X_0^\dagger \to p^\dagger$ with $p^\dagger$ the standard log points, that is, $\overline{\mathcal{M}}_{p^\dagger} = \mathcal{M}_{p^\dagger}/\mathbf{k}^* \cong \mathbb{N}$.

Definition 3.1. We call such a log smooth morphism $\pi : X_0^\dagger \to p^\dagger$ a *simple degeneration* if the underlying space X_0 is given by two smooth varieties Y_1 and Y_2 intersecting transversally along a connected smooth divisor D.

Remark 3.2. By [Ols03b], any log smooth morphism π as above which is a simple degeneration admits a canonical log structure $\tilde{\pi} : \tilde{X}^\dagger \to p^\dagger$ and a morphism $g : p^\dagger \to p^\dagger$ such that π is the pull-back of $\tilde{\pi}$ along g.

Assume that we are in the situation of Section 1.2. We fix a smooth $(d_1, \cdots, d_{l-1})$-complete intersection W of codimension $l-1$ in $\mathbb{P}^n$ cut out by $F_1, \cdots, F_{l-1}$. Let G_l be the product G_1G_2 of two homogeneous polynomials of degree a and $d_l - a$, respectively, and let F_l be a homogeneous polynomial of degree d_l.

Consider the pencil of divisors in $Z \subset W \times \mathbb{A}^1$ defined by $\{t \cdot F_l + G_l = 0\}$. Let $\pi : Z \to \mathbb{A}^1$ be the projection to the second factor.

For a general choice of $F_1, \cdots, F_l, G_1$, and G_2, there exists an open neighborhood $U \subset \mathbb{A}^1$ of 0 satisfying the following properties:

(i) The projection $\pi : \mathcal{X} := \pi^{-1}U \to U$ is a flat family of $(d_1, \cdots, d_l)$-complete intersections in $\mathbb{P}^n$.

(ii) The general fibers $\mathcal{X}_t$ are smooth.

(iii) The special fiber $\mathcal{X}_0$ is a union of a smooth $(d_1, \cdots, d_{l-1}, a)$-complete intersection X_1 and a smooth $(d_1, \cdots, d_{l-1}, d_l - a)$-complete intersection X_2.

(iv) The intersection D of X_1 and X_2 is a smooth $(d_1, \cdots, d_{l-1}, a, d_l - a)$-complete intersection.

(v) The singular locus of the total space is given by the base locus $\{F_l = 0\} \cap D$ and is of codimension one in D.

Let $\mathcal{X}^\circ$ be the complement of $\{F_l = 0\} \cap D$ in $\mathcal{X}$; it is the smooth locus of the total space $\mathcal{X}$ in the usual sense. Consider the canonical divisorial log structure $\mathcal{M}_{\mathcal{X}^\circ}$ associate with the pair $(\mathcal{X}^\circ, \partial\mathcal{X}^\circ := \pi^{-1}(0))$ and the log structure $\mathcal{M}_{\mathbb{A}^1}$ associated with $(\mathbb{A}^1, 0)$. Then we have a morphism of log schemes

$$\pi^\dagger : (\mathcal{X}^\circ, \mathcal{M}_{\mathcal{X}^\circ}) \longrightarrow (\mathbb{A}^1, \mathcal{M}_{\mathbb{A}^1})\,. \tag{3.1.1}$$

Write

$$\pi_0^\dagger : Y^\dagger \longrightarrow 0^\dagger \tag{3.1.2}$$

for the fiber over $0 \in \mathbb{A}^1$. The closure of the underlying scheme Y of $Y^\dagger$ is given by $X_1 \cup X_2$.

Lemma 3.3. *The log map $\pi^\dagger$ is a log smooth morphism of fine and saturated log schemes and the central fiber $Y^\dagger$ is a simple degeneration. In particular, the general point of D lies in the log smooth locus of $\pi^\dagger$.*

Proof. This is because the pair $(\mathcal{X}^\circ, \partial\mathcal{X}^\circ)$ over $\mathbb{A}^1$ is a simple normal crossing degeneration. $\square$

3.2 The gluing construction

Lemma 3.4. *Let $\pi : Y^\dagger \to p^\dagger$ be a simple degeneration with canonical log structures as in Definition 3.1. Let $\underline{f} : C \to Y$ be a genus zero stable map with C given by two irreducible components C_1 and C_2 glued along a node $x \in C$. Further assume $f^{-1}(D) = x$ with the same contact order c on each component. Then there is a stable log map over the underlying stable map $\underline{f}$, given by the following diagram:*

$$\begin{tikzcd} C^\dagger \arrow[r, "f"] \arrow[d] & Y^\dagger \arrow[d] \\ p^\dagger \arrow[r, "u"] & p^\dagger\,, \end{tikzcd} \tag{3.2.1}$$

such that the log structure associated with $C^\dagger \to p^\dagger$ is the canonical one as in [Kat96, Ols07]. Furthermore, on the level of characteristics, $\bar{u}^\flat : \mathbb{N} \to \mathbb{N}$ is given by multiplication by c.

Proof. This follows from the construction in [Kim10, Section 5.2.3]. $\square$

We next consider a log smooth variety $X^\dagger$ given by a smooth variety X and a smooth divisor $D \subset X$. Consider the $\mathbb{P}^1$-bundle $\mathbb{P} := \mathbb{P}(N_{D/X} \oplus \mathcal{O}_D)$ with two disjoint divisors $D_0 \cong D_\infty \cong D$ such that $N_{D_0/\mathbb{P}} \cong N_{D/X}^\vee$ and $N_{D_\infty/\mathbb{P}} \cong N_{D/X}$. Gluing $\mathbb{P}$ and X by identifying D_0 with D, we obtain a scheme Y. By [Kat96, Theorem 11.2], there is simple degeneration where the central fiber $\pi : Y^\dagger \to p^\dagger$ is a log smooth simple degeneration as in Lemma 3.4. By [GS13, Proposition 6.1], there is a log map $g : Y^\dagger \to X^\dagger$ contracting the $\mathbb{P}^1$-bundle $\mathbb{P}$ to the divisor D.

Lemma 3.5. *Consider a genus zero stable map $\underline{f} : C \longrightarrow D$ such that $C \cong \mathbb{P}^1$ and $\deg(N_{D/X})|_C = c \geqslant 0$. Then there is a log map $f : C^\dagger \to X^\dagger$ over $\underline{f}$ with a unique marking $\sigma \in C^\dagger$ with contact order c.*

Proof. Pick an arbitrary point $\sigma \in C$, and fix an isomorphism

$$(N_{D/X})|_C \cong \mathcal{O}_C(c \cdot \sigma)\,.$$

Choose a section $s \in H^0(\mathcal{O}_C(c \cdot \sigma))$ with a zero of order c at σ. Thus, the section s defines a map $\underline{f}' : C \longrightarrow \mathbb{P}$ which is tangent to D_∞ only at σ, with contact order c, and does not meet D_0. By

[Kim10, Section 5.2.3], there is a log map

$$\begin{tikzcd} C^\dagger \arrow[r, "f'"] \arrow[d] & Y^\dagger \arrow[d] \\ p^\dagger \arrow[r] & p^\dagger . \end{tikzcd}$$

Consider the morphism of log schemes $g : Y^\dagger \to X^\dagger$. Now the composition $f := g \circ f'$ defines the log map we want. ☐

LEMMA 3.6. *With the notation as above, consider a genus zero stable map $\underline{f} : C \longrightarrow X$ such that*

(i) *the curve C has two irreducible components C_1 and C_2 meeting at the node x;*
(ii) *the restriction $\underline{f}|_{C_1}$ only meets D at the node x, with contact order c_1;*
(iii) *we have $\underline{f}(C_2) \subset D$ and $\deg(\underline{f}^*(N_{D/X}))_{|C_2} = c_2$.*

Assume $c_1 + c_2 \geqslant 0$. Then there is a stable log map $f : C^\dagger/p^\dagger \to X^\dagger$ over $\underline{f}$ with a single marked point $\sigma \in C_2$ of contact order $c_1 + c_2$. Furthermore, the log structure on $C^\dagger \to p^\dagger$ is the canonical one as in [Kat96, Ols07].

Proof. We define a morphism of sheaves over C_2:

$$\mathcal{O} \oplus \mathcal{O}(-c_2) \longrightarrow \mathcal{O}(c_1)\,,$$

where the arrow $\mathcal{O} \to \mathcal{O}(c_1)$ is defined by the effective divisor $c \cdot x$, and $\mathcal{O}(-c_2) \to \mathcal{O}(c_1)$ is defined by the effective divisor $(c_1 + c_2) \cdot \sigma$. This defines a morphism $C_2 \to \mathbb{P}$ tangent to D_∞ and D_0 at σ and x with contact orders $c_1 + c_2$ and c_1, respectively. We are in the situation of Lemma 3.4. Thus, there is a stable log map f' as in (3.2.1) over the underlying map $\underline{f}$. The composition $f := f' \circ g$ yields the stable log map as in the statement. ☐

Remark 3.7. In Lemma 3.6, the marking σ can be removed if $c_1 + c_2 = 0$.

For the reader's convenience, we include the following result, which is known to the experts.

LEMMA 3.8. *Consider a genus zero log map*

$$\begin{tikzcd} C^\dagger \arrow[r, "f"] \arrow[d] & \mathcal{X}^\dagger \arrow[d] \\ S^\dagger \arrow[r] & B^\dagger , \end{tikzcd}$$

where the underlying S is a geometric point, $\mathcal{X}^\dagger \to B^\dagger$ is a log smooth family and the log structure of $B^\dagger$ over the generic point is trivial. Assume that $f^*T_{\mathcal{X}^\dagger/B^\dagger}$ is semipositive. Let f' be a general smoothing of f. Then:

(i) *The source curve of f' is irreducible.*
(ii) *The map f' only meets the locus $\partial\mathcal{X}^\dagger$ with nontrivial log structure at the marked points.*

Proof. Let $\mathfrak{K}$ be the moduli space of stable log maps, and let $\mathfrak{M}$ be the moduli space of genus zero prestable curves with its canonical log structure. Then the semipositivity of $f^*T_{\mathcal{X}^\dagger/B^\dagger}$ implies that the morphism of usual algebraic stacks

$$\mathfrak{K} \longrightarrow \mathcal{L}og_{\mathfrak{M}\times B^\dagger} \tag{3.2.2}$$

is smooth at the point $[f] \in \mathfrak{K}$; see [Che10, Section 2.5]. Here $\mathcal{L}og_{\mathfrak{M}\times B^\dagger}$ is Olsson's log stack parameterizing log structures over $\mathfrak{M} \times B^\dagger$; see [Ols03a]. By assumption, $\mathcal{L}og_{\mathfrak{M}\times B^\dagger}$ contains an open dense substack with trivial log structures. Thus, a general deformation f' satisfies the conditions in the statement; see for example [Che10, Section 3.2]. □

3.3 From Fano to log Fano via a simple degeneration

Proposition 3.9. *We use the notation of Lemma 3.3. Consider the two log Fano varieties $X_i^\dagger$ associated with (X_i, D) for $i = 1, 2$ as in Section 3.1. If $X_1^\dagger$ is separably $\mathbb{A}^1$-connected and $X_2^\dagger$ is separably $\mathbb{A}^1$-uniruled, then the general fibers of (3.1.1) are separably rationally connected.*

Proof. By assumption, we may take a very free $\mathbb{A}^1$-curve $f_1 : C_1^\dagger \to X_1^\dagger$ and a free $\mathbb{A}^1$-curve $f_2 : C_2^\dagger \to X_2^\dagger$. Write σ_i for the unique marking on $C_i^\dagger$ for $i = 1, 2$. Note that the markings of the free log map sweep out general points on the boundary divisor [KM99, Corollary 5.5(3)]. We may assume $f_1(\sigma_1) = f_2(\sigma_2)$ and $f_i(C_i^\dagger \setminus \{\sigma_i\}) \cap D_i = \emptyset$ for $i = 1, 2$.

After composing f_i with some generically étale multiple cover by rational curves ramified at σ_i, we may assume that f_1 and f_2 have the same contact orders along the common boundary. By Lemma 3.4, we may glue f_1 and f_2 along the markings, and obtain a stable log map $f : C^\dagger \to X^\dagger$ where the underlying curve C is a rational curve with one node obtained by gluing C_1 and C_2 along the markings.

Since the pull-backs of the log tangent bundles $f_1^* T_{X_1^\dagger}$ and $f_2^* T_{X_2^\dagger}$ are at least semipositive, there exists a smoothing f' of f to the general fiber of the one-parameter degeneration, by Lemma 3.8. Since $f_1^* T_{X_1^\dagger}$ is ample by assumption, a general smoothing f' is very free. □

4. Reduction to the Fano boundary

4.1 Separable $\mathbb{A}^1$-uniruledness

The following can be found in [KM99, Lemma 5.2]. For completeness, we include the proof here.

Lemma 4.1. *Let $X^\dagger$ be a log smooth scheme given by a normal crossing pair $(X, D = \sum_{i=1}^k D_i)$. Then we have an exact sequence*

$$0 \longrightarrow \mathcal{O}_{D_i} \longrightarrow T_{X^\dagger}|_{D_i} \longrightarrow T_{D_i^\dagger} \longrightarrow 0\,, \tag{4.1.1}$$

where $D_i^\dagger$ is given by the pair $(D_i, \sum_{j\neq i} D_j|_{D_i})$.

Proof. Write $Z^\dagger$ for the log scheme given by $(X, \sum_{j\neq i} D_j)$. Consider following the exact sequence over X:

$$0 \longrightarrow \Omega_{Z^\dagger} \longrightarrow \Omega_{X^\dagger} \longrightarrow \mathcal{O}_{D_i} \longrightarrow 0\,.$$

Applying $\otimes \mathcal{O}_{D_i}$ to this sequence, we have

$$0 \longrightarrow \mathrm{Tor}_1^{\mathcal{O}_X}(\mathcal{O}_{D_i}, \mathcal{O}_{D_i}) \longrightarrow \Omega_{Z^\dagger}|_{D_i} \longrightarrow \Omega_{X^\dagger}|_{D_i} \longrightarrow \mathcal{O}_{D_i} \longrightarrow 0\,.$$

Note that $\mathrm{Tor}_1^{\mathcal{O}_X}(\mathcal{O}_{D_i}, \mathcal{O}_{D_i}) = N_{D_i/X}^\vee$. Now the statement follows from taking the dual of the last exact sequence. □

Lemma 4.2. *We use the notation of Lemma 4.1. If there exists a $D_i^\dagger$ that is separably $\mathbb{A}^1$-uniruled with $\deg f^* N_{D_i} > 0$ for some free $\mathbb{A}^1$-curve $f : Z^\dagger \to D_i^\dagger$, then $X^\dagger$ is separably $\mathbb{A}^1$-uniruled.*

When D is a smooth irreducible ample divisor which is separably uniruled, $X^\dagger$ is separably $\mathbb{A}^1$-uniruled.

Proof. We will give a proof of the first statement. The second statement can be proved similarly.

By the assumption and log deformation theory, we may choose a free $\mathbb{A}^1$-curve $f' : Z^\dagger \to D_i^\dagger$ such that

(i) the underlying source curve $Z \cong \mathbb{P}^1$ is irreducible with a unique marking $\sigma \in Z$;
(ii) we have $f'(Z) \not\subset D_j$ for any $j \neq i$;
(iii) $\deg f'^* N_{D_i|X} > 0$.

For each j, consider another log scheme $X_j^\dagger$ given by the pair (X, D_j). By Lemma 3.5, we can lift f' to a genus zero stable log map $f_i'' : C^\dagger/S^\dagger \to X_i^\dagger$ with the unique marking σ. Since the image of f_i'' is not contained in D_j for any $j \neq i$, we obtain a stable log map $Z^\dagger \to X_j^\dagger$ with the same underlying map given by f' and a unique marking σ, possibly having trivial contact order. Consider the composition $f_j'' : C^\dagger \longrightarrow Z^\dagger \longrightarrow X_j^\dagger$. Now the product

$$f := \prod_{j=1}^{k} f_j'' : C^\dagger \longrightarrow X_1^\dagger \times_X \cdots \times_X X_k^\dagger \cong X^\dagger$$

defines an $\mathbb{A}^1$-curve in $X^\dagger$. Using Lemma 4.1, we can check that f is a free $\mathbb{A}^1$-curve. This finishes the proof. $\square$

4.2 Separable $\mathbb{A}^1$-connectedness

The goal of this section is to prove the following result.

PROPOSITION 4.3. *Let $X^\dagger$ be a general log Fano $(d_1, \cdots, d_l; d_b)$-complete intersection given by the pair (X, D) as in Section 1.2. If D is separably rationally connected and* $\operatorname{char} \mathbf{k} \nmid d_b$, *then $X^\dagger$ is separably $\mathbb{A}^1$-connected.*

Proof. Choose a very free rational curve $\underline{f}_1 : C_1 \cong \mathbb{P}^1 \to D$. Let σ, σ_1 be two general points on C_1. Since $\operatorname{char} \mathbf{k} \nmid d_b$, we may choose a log free line $f_2 : C_2^\dagger \to X^\dagger$ constructed in Proposition 2.2 with the unique marking σ_2 having image $\underline{f}_1(\sigma_1)$. By Lemma 3.6, we may glue $\underline{f}_1$ and $\underline{f}_2$ by identifying σ_1 and σ_2, and obtain a stable log map $f : C^\dagger \to X^\dagger$ with one marking σ and one node p.

If we restrict (4.1.1) to C_1, there are two possibilities:

(i) The bundle $T_{X^\dagger}|_{C_1}$ is ample.
(ii) The bundle $T_{X^\dagger}|_{C_1}$ is a trivial extension of $T_D|_{C_1}$ by $\mathcal{O}_{C_1}$.

In the first case, a general smoothing f is very free by Lemma 3.8. In the second case, $T_{X^\dagger}|_{C_1}$ is only semipositive.

Consider the composition

$$T_{C_2^\dagger}|_p \xrightarrow{\mathrm{d}f_2} T_{X^\dagger}|_p \xrightarrow{\delta} T_D|_p\,. \tag{4.2.1}$$

LEMMA 4.4. *The push-forward morphism $\mathrm{d}f_2$ is injective when* $\operatorname{char} \mathbf{k} \nmid d_b$.

Proof. It suffices to show that the pull-back morphism $(\mathrm{d}f_2)^\vee : \Omega_{X^\dagger}|_p \to \Omega_{C^\dagger}|_p$ is surjective. We check this using a local computation. Locally at p, there is a log one-form $\mathrm{d}g/g$ where g is the defining equation of the boundary. Since the image of C_2 is a log free line, $(\mathrm{d}f_2)^\vee(\mathrm{d}g/g) = d_b \cdot \mathrm{d}t/t \neq 0$. $\square$

Lemma 4.5. *The composite morphism (4.2.1) is the zero morphism.*

Proof. Applying Lemma 4.1 to both $C_2^\dagger$ and $X^\dagger$, and restricting to σ, we have the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{k}_{\sigma_2} \arrow[r, "\cong"] \arrow[d] & T_{C_2^\dagger}|_{\sigma_2} \arrow[r] \arrow[d] & 0 & \\ 0 \arrow[r] & \mathcal{O}_D|_{\sigma_2} \arrow[r] & T_{X^\dagger}|_{\sigma_2} \arrow[r] & T_D|_{\sigma_2} \arrow[r] & 0\,. \end{tikzcd}$$

The statement then follows. □

Let E be the codimension one vector subspace in $T_{X^\dagger}|_p$ which corresponds to T_D. To make a log very free curve, it suffices to increase the positivity outside E. By Proposition 2.2, the splitting type of $T_{X^\dagger}|_{C_2}$ is $\mathcal{O}(1)^{\oplus(n+1-e)} \oplus \mathcal{O}^{\oplus(e-l-1)}$. Let E' be the canonical subspace of $T_{X^\dagger}|_p$ which corresponds to the factor $\mathcal{O}(1)^{\oplus(n+1-e)}$. By Lemma 4.4, E' contains the log tangent direction $T_{C_2^\dagger}|_p$.

By Lemma 4.5, E' as a vector subspace in $f^*T_{X^\dagger}|_p$ is contained in the kernel of $T_{X^\dagger}|_p \to T_D|_p$. Since E is of codimension one, the two vector subspaces E' and E span $T_{X^\dagger}|_p$.

Since f is unobstructed with canonical log structure $C^\dagger \to p^\dagger$ on the source log curve, this implies that the composition

$$\mathfrak{K} \longrightarrow \mathcal{L}og_{\mathfrak{M}} \longrightarrow \mathfrak{M}$$

is smooth at the point $[f]$. Here the first arrow is given by (3.2.2) with $B^\dagger$ a geometric point with trivial log structure. We may thus take a general smoothing of f with smooth total space. Proposition 4.3 then follows from Proposition 4.9 below. □

Proof of Theorem 1.9. By adjunction, D is Fano, and hence separably rationally connected in characteristic zero. We may then choose a very free rational curve $\underline{f} : C \to D$ through general points of D. Now the theorem is proved by gluing $\underline{f}$ with a free $\mathbb{A}^1$-curve in (X, D) with sufficiently large intersection number with D, and applying the same argument as in Proposition 4.3. □

Proof of Corollary 1.10. By Lemma 4.2 and Theorem 1.9, it suffices to show that there exists a free rational curve $f : \mathbb{P}^1 \to D$ such that $\deg f^*N_D > 0$. Indeed, by the adjunction formula, D is Fano, hence rationally connected. Let $E \subset D$ be the effective divisor determined by N_D. A very free rational curve passing through a point in E but not lying on E will do the job. □

4.3 A result from the theory of elementary transformations

Construction 4.6. Let C be the union of two irreducible rational curves C_1 and C_2 glued at a node p. Let $q : \mathcal{C} \to T$ be a smoothing of C with C the fiber over $0 \in T$. Assume that the total space $\mathcal{C}$ is a smooth surface. Let s_1 and s_2 be two sections of q which specialize to two distinct points y_1 and y_2, respectively, on C_1. Consider a locally free sheaf $\mathcal{E}$ of rank r on $\mathcal{C}$, satisfying the following properties:

- (i) The sheaf $\mathcal{E}|_{C_1}$ is isomorphic to $\mathcal{O} \oplus \mathcal{F}$, where $\mathcal{F}$ is a positive subbundle.
- (ii) We have $\mathcal{E}|_{C_2} \cong \mathcal{T} \oplus \mathcal{O}^{\oplus r-k}$, where $1 \leqslant k \leqslant r$ and $\mathcal{T}$ is positive.
- (iii) Let E be the canonical codimension one subspace of $\mathcal{E}|_p$ which corresponds to $\mathcal{F}$ and let E' be the canonical subspace of $\mathcal{E}|_p$ which corresponds to $\mathcal{T}$. Then the subspaces E' and E span $\mathcal{E}|_p$.

Consider the following composition:

$$r : \mathcal{E}^\vee \longrightarrow \mathcal{E}^\vee|_{C_2} \longrightarrow \mathcal{T}^\vee . \tag{4.3.1}$$

Clearly r is surjective. Let $\mathcal{K}^\vee$ be the kernel of r, that is, the elementary transformation of $\mathcal{E}^\vee$ along $\mathcal{T}^\vee$. Consider the induced exact sequence

$$0 \longrightarrow \mathcal{K}^\vee \longrightarrow \mathcal{E}^\vee \longrightarrow \mathcal{T}^\vee \longrightarrow 0 . \tag{4.3.2}$$

Dualizing this short exact sequence over $\mathcal{C}$, we get a long exact sequence

$$0 \longrightarrow \hom_{\mathcal{O}_\mathcal{C}}(\mathcal{T}^\vee, \mathcal{O}_\mathcal{C}) \longrightarrow \mathcal{E} \longrightarrow \mathcal{K} \longrightarrow \underline{\mathrm{Ext}}^1_{\mathcal{O}_\mathcal{C}}(\mathcal{T}^\vee, \mathcal{O}_\mathcal{C}) \longrightarrow 0 .$$

The first term vanishes because it is the dual of a torsion sheaf. The last term is isomorphic to $\mathcal{T} \otimes_{\mathcal{O}_{C_2}} \mathcal{O}_{C_2}(C_2)$ by [Eis95, Exercise A3.46 b], and

$$\mathrm{Ext}^1_{\mathcal{O}_\mathcal{C}}(\mathcal{O}_{C_2}, \mathcal{O}_\mathcal{C}) \cong \mathcal{O}_{C_2}(C_2) .$$

Thus we obtain a short exact sequence

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{K} \longrightarrow \mathcal{T} \otimes_{\mathcal{O}_{C_2}} \mathcal{O}_{C_2}(C_2) \longrightarrow 0 . \tag{4.3.3}$$

Lemma 4.7. *We have* $h^1(C_2, \mathcal{K}|_{C_2}(-p)) = 0$.

Proof. Restricting the short exact sequence (4.3.2) to C_2, applying the functor $\underline{\mathrm{Hom}}_{\mathcal{O}_{C_2}}(* , \mathcal{O}_{C_2})$ and combining with (4.3.3), we obtain

$$0 \longrightarrow \mathcal{T} \longrightarrow \mathcal{E}|_{C_2} \longrightarrow \mathcal{K}|_{C_2} \longrightarrow \mathcal{T}(-p) \longrightarrow 0 .$$

The quotient bundle $\mathcal{E}|_{C_2}/\mathcal{T}$ is a trivial vector bundle and the last term of the exact sequence is isomorphic to $\mathcal{T}(-p)$. In particular, we have

$$0 \longrightarrow \mathcal{O}_{C_2}(-p)^{\oplus(r-k)} \longrightarrow \mathcal{K}|_{C_2}(-p) \longrightarrow \mathcal{T}(-2p) \longrightarrow 0 .$$

The lemma follows from the vanishing of the H^1 of the first and third terms of this sequence. ☐

Lemma 4.8. *We have* $h^1(C_1, \mathcal{K}|_{C_1}(-y_1 - y_2)) = 0$.

Proof. Restricting the short exact sequence (4.3.2) to C_1, we get

$$\mathcal{K}^\vee|_{C_1} \longrightarrow \mathcal{E}^\vee|_{C_1} \longrightarrow \mathcal{T}^\vee|_{C_1} \longrightarrow 0 .$$

This sequence is also left exact. Indeed, since $\mathcal{T} \otimes_{\mathcal{O}_{C_2}} \mathcal{O}_{C_2}(C_2)|_{C_1}$ is torsion, by restricting (4.3.3) to C_1 and taking the dual over C_1, we have the injection from $\mathcal{K}^\vee|_{C_1}$ to $\mathcal{E}^\vee|_{C_1}$.

In other words, the vector bundle $\mathcal{K}|_{C_1}$ is the elementary transformation of $\mathcal{E}|_{C_1}$ along p with the specific subspace E'. By condition (iii) of the construction, E' does not lie in $\mathcal{F}$ at p. This implies that $\mathcal{K}$ is ample on C_1. The statement follows. ☐

Proposition 4.9. *With the same notation and constructions as above, the restriction of* $\mathcal{E}$ *to a general fiber* $\mathcal{C}_t$ *is positive.*

Proof. By the construction, we know that $\mathcal{K}|_{\mathcal{C}_t}$ is isomorphic to $\mathcal{E}|_{\mathcal{C}_t}$. Since $\mathcal{K}$ is locally free on $\mathcal{C}$, it is flat over T. By upper semicontinuity, it suffices to show that $h^1(C, \mathcal{K}(-y_1 - y_2)) = 0$. This follows from the two lemmas above and the restrictions of the short exact sequence. ☐

Acknowledgements

The authors would like to thank Dan Abramovich for sharing his ideas and giving many helpful suggestions on the preliminary versions of this paper. They would like to thank Chenyang Xu for suggesting this collaboration and for useful discussions. The authors are grateful to Johan de Jong and Jason Starr for useful discussions, and to the anonymous referee for his/her detailed comments and suggestions on the manuscript.

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