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$\mathbb{A}^1$ curves on log K3 surfaces

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ABSTRACT

In this paper, we study $\mathbb{A}^1$ curves on log K3 surfaces. We classify all genuine log K3 surfaces of type II which admit countably infinite $\mathbb{A}^1$ curves.

1. Introduction

From the point of view of birational geometry, $\mathbb{A}^1$ curves play the roles for log varieties as rational curves do for projective varieties. However, much less is known in the log world, even in two dimensional case. $\mathbb{A}^1$ curves on log varieties with negative log Kodaira dimension are studied in [13,9,3,4,18]. We are interested in $\mathbb{A}^1$ curves on log surfaces of

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Calabi–Yau type, namely, log K3 surfaces. They play important roles in the recent work of M. Gross, P. Hacking, S. Keel and M. Kontsevich on mirror symmetry for log K3 surfaces [5,6].

Inspired by the recent progress on the existence of countably many rational curves on a projective K3 surface ([1,2] and [11]), we propose the following question studying $\mathbb{A}^1$ curves on log K3 surfaces classified by S. Iitaka [8] and D.Q. Zhang [17].

Question 1.1. *For which log K3 surfaces (X, D), are there infinitely many $\mathbb{A}^1$ curves on $X \backslash D$?*

A *log K3* surface, in the sense of Iitaka, is a log smooth projective pair (X, D) satisfying $h^0(K_X + D) = 1$ and $\kappa(X, D) = q(X, D) = 0$. According to Iitaka's classification, there are two types of log K3's:

Type I: X is birational to a projective K3 surface;
Type II: X is a smooth projective rational surface.

In this paper, we are mainly interested in a special class of log K3 surfaces:

Definition 1.2. A *genuine log K3* surface is a log smooth projective surface pair (X, D) such that

(1) $K_X + D = 0$ in $\operatorname{Pic}(X)$;
(2) $q(X, D) = h^0(\Omega^1_X(\log D)) = 0$.

In Iitaka's classification, genuine log K3 surfaces serve as the building blocks of log K3 surfaces. Of course, a genuine log K3 surface of type I is simply a projective K3 surface without boundary. It has been proved by J. Li and C. Liedtke that there are infinitely many rational curves on almost every projective K3 surface X (provided that $\operatorname{rank}_{\mathbb{Z}} \operatorname{Pic}(X)$ is odd or $\operatorname{rank}_{\mathbb{Z}} \operatorname{Pic}(X) \geq 5$ or X has an elliptic fiberation) [11]. So we have a nearly complete answer to Question 1.1 for genuine log K3's of type I. In this paper, we study this question for genuine log K3's of type II.

Since the existence of $\mathbb{A}^1$ curves is essentially a property of the open part $X \backslash D$ of a log variety (X, D), we consider (X_1, D_1) and (X_2, D_2) to be *log isomorphic* if there exists a birational map $f : X_1 \dashrightarrow X_2$ inducing an isomorphism $f : X_1 \backslash D_1 \cong X_2 \backslash D_2$ and we call such f a *log isomorphism*. For genuine log K3's of type II, we have the following classification under log isomorphisms.

Theorem 1.3. *Every genuine log K3 surface (X, D) of type* II *is log isomorphic to one of the following genuine log K3 surfaces $(\widehat{X}, \widehat{D})$:*

C0. *$\widehat{D}$ is a smooth elliptic curve;*
C1. *$\widehat{D}$ is a nodal rational curve;*
C2. *$\widehat{D} = \widehat{D}_1 + \widehat{D}_2 + ... + \widehat{D}_n$ is a circular boundary (see below) satisfying $\widehat{D}_i^2 \le -2$ for $i \ne 1$ and $\widehat{D}_1^2 \ne 0, -1$;*
C3. *$\widehat{D} = \widehat{D}_1 + \widehat{D}_2$ is a circular boundary satisfying $\widehat{D}_1^2 \ne -1$ and $\widehat{D}_2^2 = 0$;*
C4. *$\widehat{D} = \widehat{D}_1 + \widehat{D}_2$ is a circular boundary satisfying $\widehat{D}_1^2 > 0$ and $\widehat{D}_2^2 > 0$.*

We have a complete answer to Question 1.1 for genuine log K3 surfaces (X, D) of Iitaka type II by our main theorem:

Theorem 1.4 *($\mathbb{A}^1$ curves on genuine log K3's of Iitaka type II). Let (X, D) be a genuine log K3 surface of type II. Then there are countably many $\mathbb{A}^1$ curves in $X \backslash D$ if and only if (X, D) is log isomorphic to one of C0–C3 in Theorem 1.3.*

It is relatively easier to prove the existence of infinitely many $\mathbb{A}^1$ curves on (X, D) of type C0, C1 and C3 compared with C2. For (X, D) of type C2, we can contract $D_2 + D_3 + ... + D_n$ to obtain a *log del Pezzo* surface $\overline{X}$, i.e., a projective surface with at worst log terminal singularities and $-K_{\overline{X}}$ ample. Here is where the celebrated theorem of Keel–McKernan comes in: $\overline{X}$ is rationally connected [9]. We will use this to show that there are infinitely many $\mathbb{A}^1$ curves in $\overline{X} \backslash \overline{D}$.

As suggested to us by David McKinnon, our construction of $\mathbb{A}^1$ curves on log K3 surfaces over number fields actually produce an infinite sequence of $\mathbb{A}^1$ curves defined over number fields of increasing degrees over $\mathbb{Q}$.

Theorem 1.5. *For a genuine log K3 surface (X, D) over $\overline{\mathbb{Q}}$ where D is either a smooth elliptic curve or a rational curve with one node, there does not exist a number field $k \subset \overline{\mathbb{Q}}$ such that every $\mathbb{A}^1$ curve in $X \backslash D$ is defined over k.*

Note that for a log K3 surface (X, D) over $\overline{\mathbb{Q}}$, every $\mathbb{A}^1$ curve in $X_{\mathbb{C}} \backslash D_{\mathbb{C}}$ is automatically defined over $\overline{\mathbb{Q}}$ due to rigidity (see Lemma 3.4).

The similar statement for rational curves on K3 surfaces over number fields is expected but not known, to the best of our knowledge. Although J. Li and C. Liedtke proved that almost all K3 surfaces over number fields have infinitely many rational curves, it is not clear that the rational curves they produced lie over an ascending chain of number fields.

The paper is organized as follows. In §3, we deal with genuine log K3 surfaces (X, D) of type C0 and C1 and prove the existence of infinitely many $\mathbb{A}^1$ curves and Theorem 1.5 for such (X, D). Theorem 1.3 is proved in §2 and our main Theorem 1.4 is proved in §4. In §5, we put our results under the framework of Iitaka's classification of log K3 surfaces and give examples of genuine log K3 surfaces that do not have infinitely many $\mathbb{A}^1$ curves.

1.1. Remarks on Question 1.1

Question 1.1 is very difficult in general. For example, we do not know the case when (X, D) is obtained from the blowup $\pi : X \to S$ of a K3 surface S at finitely many points Σ with D the exceptional divisor of π; finding $\mathbb{A}^1$ curves in $X \backslash D$ amounts to finding rational curves on S missing all but one point in Σ, which turns out to be a surprisingly difficult problem. An affirmative answer would generalize the theorem of Li–Lietke [11]. On the other hand, there are log K3 surfaces with no log rational curves at all. For example, let X be a Kummer K3 and let D be the disjoint union of 16 (-2)-curves. Then (X, D) is a log K3 with no $\mathbb{A}^1$ curves because $X \backslash D$ has an étale cover by an abelian surface deleting 16 points. Also there are many examples of genuine log K3 surfaces of type II without infinitely many $\mathbb{A}^1$ curves (see §5). This suggests that the condition on the vanishing of log irregularity is too weak to ensure the existence of $\mathbb{A}^1$ curves.

1.2. Convention and terminology

We work exclusively over algebraically closed fields of characteristic 0. Throughout the paper, “countable” means “countably infinite”.

A log pair (X, D) means a variety X with a reduced Weil divisor D. Let U be its interior $X - D$. We say that (X, D) is *log smooth* if X is smooth and D is a normal crossing divisor. A log pair is projective if the ambient variety is projective.

For a log smooth pair (X, D), we use $\kappa(X, D)$ to denote the logarithmic Kodaira dimension and $q(X, D)$ to denote the logarithmic irregularity, i.e., $q(X, D) = h^0(X, \Omega^1_X(\log D))$. They only depend on the interior of the pair.

An $\mathbb{A}^1$ (or log rational) curve C° in $X \backslash D$ is a quasi-projective curve whose normalization is $\mathbb{A}^1$. Alternatively, the closure C of C° in X is a rational curve satisfying that $\nu^{-1}(D)$ consists of at most one point for the normalization $\nu : C^\nu \to X$ of C.

It is easy to see that a genuine log K3 surface (X, D) of type II must be one of the following:

(1) D is a smooth elliptic curve.
(2) D is a rational curve with one node.
(3) D is a union of smooth rational curves with simple normal crossings (snc) whose dual graph is a “circle”, called a “circular boundary” by Iitaka. That is, we have $D = D_1 + D_2 + ... + D_n$ such that

$$\begin{aligned} D_i(D - D_i) &= 2 \text{ for all } i \\ D_i D_j &= 0 \text{ for } i - j \not\equiv 0, \pm 1 \pmod{n}. \end{aligned} \tag{1.1}$$

We call such D a *circular boundary* of type $(\lambda_1, \lambda_2, ..., \lambda_n)$ if $D_i^2 = \lambda_i$.

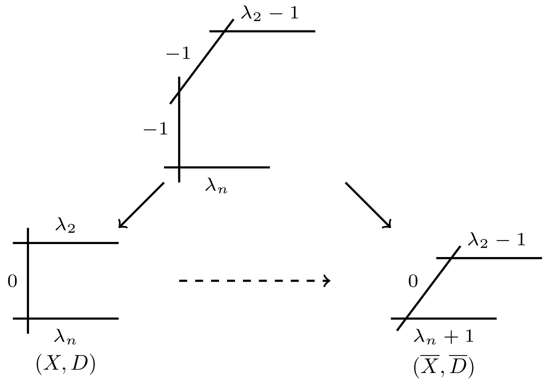

Fig. 1. A pivot $\pi : (X, D) \dashrightarrow (\overline{X}, \overline{D})$ at D_1.

For a log surface (X, D) with X smooth and D a nc divisor, a *canonical blowup* $f : (\widehat{X}, \widehat{D}) \to (X, D)$ is the blowup of X at a singular point $p \in D_{\text{sing}}$ of D with $\widehat{D} = f^{-1}(D)$ and a *canonical blowdown* $g : (X, D) \to (\overline{X}, \overline{D})$ is the contraction of a (-1)-curve contained in D with $\overline{D} = g_* D$.

2. Proof of Theorem 1.3

The key construction here is a "pivot operation", which is also needed in the proof of our main theorem.

Proof Theorem 1.3. We use the notation $\mu(G)$ to denote the number of irreducible components in a curve G. We will argue by induction on $\mu(D)$.

If $\mu(D) = 1$, D must be a smooth elliptic curve or a nodal rational curve and we have C0 or C1.

Suppose that $D = D_1 + D_2 + ... + D_n$ is a circular boundary of type $(\lambda_1, \lambda_2, ..., \lambda_n)$ with $D_i^2 = \lambda_i$. If D contains a (-1)-curve D_i, we simply let $\pi : X \to \overline{X}$ be the contraction of D_i. Obviously, π is a log isomorphism and we have reduced $\mu(D)$ by 1. Suppose that $\lambda_i \neq -1$ for all i.

Suppose that $\mu(D) = 2$. If $\lambda_1 = 0$ or $\lambda_2 = 0$, we have C3. Suppose that $\lambda_i \neq 0$. If $\lambda_1 \leq -2$ or $\lambda_2 \leq -2$, we have C2. Otherwise, $\lambda_1, \lambda_2 > 0$ and we have C4.

Suppose that $\mu(D) = n \geq 3$. If $\lambda_i \leq -2$ for all but one i, we are done. Let us assume that there are at least two nonnegative λ_i's.

Suppose that one of λ_i's vanishes, say $\lambda_1 = 0$. We have a log isomorphism $\pi : (X, D) \dashrightarrow (\overline{X}, \overline{D})$ composed of a blowup of X at $D_1 \cap D_2$ followed by a blowdown of the proper transform of D_1. On $\overline{X}$, we have $\overline{D}_n^2 = \lambda_n + 1$, $\overline{D}_1^2 = 0$ and $\overline{D}_2^2 = \lambda_2 - 1$ if $n \geq 3$. We call such π a *pivot* at D_1 (see Fig. 1). When $n \geq 3$, applying a sequence of pivot operations at D_1, we arrive at (X, D) with D a circular boundary of type $(0, -1, \lambda_3, ..., \lambda_{n-1}, \lambda_n + \lambda_2 + 1)$; we then contract D_2, which will reduce $\mu(D)$ by 1.

Finally, we have the remaining case that $\lambda_i \neq 0, -1$ for all i, at least two of λ_i's are positive and $\mu(D) \geq 3$. Suppose that $\lambda_i, \lambda_j > 0$ for some $i \neq j$. Since [8, Lemma 2, p. 682]

$$q(X, D) = \dim_{\mathbb{Q}} \ker \left(\bigoplus_{i=1}^{n} \mathbb{Q} D_i \to H^2(X, \mathbb{Q}) \right) \tag{2.1}$$

and $q(X, D) = 0$ as (X, D) is a log K3 surface, $D_1, D_2, ..., D_n$ must be linearly independent in $H^2(X, \mathbb{Q})$. In particular, D_i and D_j are linearly independent in $H^2(X, \mathbb{Q})$. Combining with the Hodge index theorem [12, Lem. 1.10.2] and X is rational, we see that

$$\det \begin{bmatrix} D_i^2 & D_i D_j \\ D_i D_j & D_j^2 \end{bmatrix} = \det \begin{bmatrix} \lambda_i & D_i D_j \\ D_i D_j & \lambda_j \end{bmatrix} < 0. \tag{2.2}$$

It follows that $D_i D_j \geq 2$. And since the dual graph of $D_1, D_2, ..., D_n$ is a circle, we conclude that D_i and D_j are the only components of D, i.e., $D_i D_j = 2$ and $n = 2$. Contradiction. □

3. Irreducible boundary case

We are going to prove the following result in this section.

Theorem 3.1. *For a genuine log K3 surface (X, D) of type II where D is either a smooth elliptic curve or a rational curve with one node, there are countably many $\mathbb{A}^1$ curves in $X \backslash D$.*

Namely, we will prove that there are countably many $\mathbb{A}^1$ curves on a genuine log K3 of type C0 or C1.

Let us first revisit the following theorem of Geng Xu [16]:

Theorem 3.2 *(G. Xu). Given a smooth cubic curve D in $\mathbb{P}^2$, there are countably many rational curves in $\mathbb{P}^2$ meeting D set-theoretically at a unique point.*

Sketch of Xu's Proof. It is easy to show that there are at most countably many rational curves meeting D at a unique point. Roughly, if there is a complete one-parameter family of such rational curves, some fiber of the family must contain D, which is impossible.

Let $V_{A,g}$ be the Severi variety of integral curves in $|A|$ of genus g and $\overline{V}_{A,g}$ be its closure in $|A| = \mathbb{P}H^0(A)$. It is well known that

$$\dim V_{A,0} = A.D - 1 = a - 1. \tag{3.1}$$

The key to produce infinitely many such rational curves is the following observation: For every ample divisor A on $X = \mathbb{P}^2$, there exists a point $p \in D$ such that

$$ap = i_D^* A \text{ in } \operatorname{Pic}(D) \text{ and } mp \notin i_D^* \operatorname{Pic}(X) \text{ for all } 0 < m < a \tag{3.2}$$

where $a = A.D$, i_D is the embedding $D \hookrightarrow X$ and $i_D^* : \operatorname{Pic}(X) \to \operatorname{Pic}(D)$ is the pullback between the Picard groups of X and D.

Let $\Lambda \subset \mathbb{P}H^0(A)$ be the subvariety consisting of $C \in |A|$ such that C meets D at p with multiplicity a. Then Λ is a linear subspace of $\mathbb{P}H^0(A)$ of codimension $a - 1$. So $\overline{V}_{A,0} \cap \Lambda \neq \emptyset$. Let $C \in \overline{V}_{A,0} \cap \Lambda$. Then every component of C is rational and hence $D \not\subset C$. So C meets D properly at p with multiplicity a. By our choice of p, C must be integral. □

Note that the rational curves meeting D set-theoretically at a single point are not necessarily $\mathbb{A}^1$ curves in $X \backslash D$: for $C \in \overline{V}_{A,0} \cap \Lambda$ in Xu's proof, there is no guarantee that $\nu^{-1}(p)$ consists of a single point on the normalization C^ν of C. Indeed, the computation of the corresponding Gromov–Witten invariants suggests that $\mathbb{A}^1$ curves form a proper subset of $\overline{V}_{A,0} \cap \Lambda$ [15].

So we need to adapt Xu's argument to $\mathbb{A}^1$ curves. There are two main ingredients of Xu's argument. One is (3.1), which guarantees that there are "sufficiently many" rational curves on X. The other is (3.2). His argument can be described by the phrase "bend-and-not-break": as he bends the rational curves in $V_{A,0}$ to meet D at p with multiplicity a, the condition (3.2) guarantees that the resulting curves do not break. Both (3.1) and (3.2) are also crucial to our argument. We have the following weak generalization of (3.2).

Lemma 3.3. *Let D be a smooth elliptic curve or a nodal rational curve of arithmetic genus $p_a(D) = 1$ on a projective variety X with the property that $i_D^* \operatorname{Pic}(X)$ is finitely generated over $\mathbb{Z}$. For every $A \in \operatorname{Pic}(X)$ with $a = AD \in \mathbb{Z}^+$, there exists a point $p \in D_{\mathrm{sm}}$ satisfying*

$$\begin{aligned} &ap = i_D^* A \text{ in } \operatorname{Pic}(D) \text{ and} \\ &mp \notin G = i_D^* \operatorname{Pic}(X) \text{ for all } m \in \mathbb{Z}^+ \text{ and } m < \sqrt{\frac{a}{|G_{\mathrm{tors}}|}} \end{aligned} \tag{3.3}$$

where G_{tors} is the torsion part of $G = i_D^* \operatorname{Pic}(X)$ and D_{sm} is the smooth locus of D.

Proof. Since torsions of all orders exist in $\operatorname{Pic}(D)$, we can find two points p_1 and p_2 on D_{sm} such that $ap_1 = ap_2 = i_D^* A$ and $p_1 - p_2$ is torsion in $\operatorname{Pic}(D)$ of order exactly a. Suppose that (3.3) fails for both p_i. Then there exist positive integers k_1 and k_2 such that $k_i^2 l < a$ and $k_i p_i \in G$ for $i = 1, 2$, where $l = |G_{\mathrm{tors}}|$. Then $k_1 k_2 l < a$ and $k_1 k_2 (p_1 - p_2) \in G$. On the other hand, $k_1 k_2 (p_1 - p_2)$ is torsion of order $\geq a/(k_1 k_2) > l$. Contradiction. □

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. It is well known that there are at most countably many $\mathbb{A}^1$ curves in $X\backslash D$ if $K_X + D$ is pseudo-effective (also see Lemma 3.4 below).

Since $K_X + D = 0$, $E.D = 1$ for every (-1)-curve $E \subset X$. So if we contract a (-1)-curve E with $g : X \to \overline{X}$, we see that $K_{\overline{X}} + \overline{D} = 0$, $\overline{D} = g_* D$ remains a smooth elliptic or nodal rational curve and $(\overline{X}, \overline{D})$ remains a genuine log K3. By contracting a sequence of (-1)-curves, we arrive at a minimal rational surface $\overline{X}$. That is, there exists a birational morphism $g : X \to \overline{X}$ with $\overline{D} = g_* D$ such that $\overline{X}$ is a minimal rational surface and $\overline{D} \in |-K_{\overline{X}}|$ is a smooth elliptic or rational nodal curve. By classification of surfaces, $\overline{X}$ must be $\mathbb{P}^2$ or a Hirzebruch surface $\mathbb{F}_\beta = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(\beta))$ over $\mathbb{P}^1$ for some $\beta \neq 1$. And since $|-K_{\overline{X}}|$ has an irreducible member, $\overline{X}$ must be one of $\mathbb{P}^2, \mathbb{F}_0$ and $\mathbb{F}_2$. Let us replace (X, D) by $(\overline{X}, \overline{D})$.

One key step of our proof is to find a fiberation $f : X \to \mathbb{P}^1$ whose general fibers are $\mathbb{P}^1$. Such fiberation obviously exists for $X \cong \mathbb{F}_0$ or $\mathbb{F}_2$ but does not for $X \cong \mathbb{P}^2$.

When $X \cong \mathbb{P}^2$, we let $\pi : \widehat{X} \to X$ be the cyclic triple cover of X ramified over D and let $\widehat{D} = \pi^{-1}(D)$. Clearly, if there are infinitely many $\mathbb{A}^1$ curves in $\widehat{X}\backslash\widehat{D}$, the same holds for $X\backslash D$. Note that $\widehat{X}$ is a smooth cubic surface when D is smooth and a cubic surface with an A_2 singularity when D is a nodal cubic. Also note that the node of $\widehat{D}$ is the same as the A_2 singularity of $\widehat{X}$, i.e., $\widehat{X}_{\text{sing}} = \widehat{D}_{\text{sing}}$.

It is easy to see that such a cubic surface $\widehat{X}$ admits a fiberation $f : \widehat{X} \to \mathbb{P}^1$ with general fibers $\mathbb{P}^1$. Let $L \subset X$ be a tri-tangent line to D at a smooth point; then $\pi^{-1}(L) = E_1 + E_2 + E_3$ is the union of three (-1)-curves $E_i \subset \widehat{X}$ meeting at one point and $|E_1 + E_2|$ is a pencil which gives us the map $f : \widehat{X} \to \mathbb{P}^1$. Furthermore, if we choose another tri-tangent line $L' \subset X$ to D with $\pi^{-1}(L') = E_1' + E_2' + E_3'$, then there exists one E_i' intersecting $E_1 + E_2$ only at one point. Such E_i' gives a section of f disjoint from the A_2 singularity. Let us replace (X, D) by $(\widehat{X}, \widehat{D})$.

In summary, we have a morphism $f : X \to \mathbb{P}^1$ whose general fibers are $\mathbb{P}^1$ and X is $\mathbb{F}_0$, $\mathbb{F}_2$, a smooth cubic surface or a cubic surface with an A_2 singularity. We may further choose a section C of f disjoint from X_{sing}.

Let F be a fiber of f. Since $C \cap X_{\text{sing}} = \emptyset$, $C + mF$ is Cartier for all $m \in \mathbb{Z}$. Let us consider $\Gamma \in |C + mF|$ for $m >> 1$.

When $C + mF$ is big and nef, we observe that the map

$$H^0(\mathcal{O}_X(C + mF)) \twoheadrightarrow H^0(\mathcal{O}_D(C + mF)) \tag{3.4}$$

is a surjection since

$$h^1(-D + C + mF) = h^1(K_X + C + mF) = h^1(-C - mF) = 0 \tag{3.5}$$

by Kawamata–Viehweg vanishing. Therefore, for $m >> 1$ and every $p \in D_{\text{sm}}$ satisfying $ap = C + mF$ in $\text{Pic}(D)$, there exists $\Gamma \in |C + mF|$ such that Γ meets D at the unique point p. Since Γ does not pass through the node of D if D is nodal and $X_{\text{sing}} = D_{\text{sing}}$, the section Γ does not pass through the singularity of X.

Since Γ is a member of the linear series $|C + mF|$, it consists of a section R of $f : X \to \mathbb{P}^1$ and a union of components $F_1, F_2, \dots, F_b$ contained in the fibers of f with certain multiplicities in Γ, i.e.,

$$\Gamma = R + m_1 F_1 + m_2 F_2 + \cdots + m_b F_b.$$

For every F_i, since

$$2p_a(F_i) - 2 = F_i^2 - F_i D < 0,$$

F_i has to be a smooth rational curve. Since Γ is away from the singularity of X, R is smooth. So Γ is supported on a union of smooth rational curves.

By Lemma 3.3, there exists p satisfying (3.3). So there exists a curve $\Gamma_m \in |C + mF|$ such that Γ_m meets D only at a point p satisfying (3.3). Let Γ'_m be an irreducible component of Γ. Then Γ'_m is a smooth rational curve meeting D at the unique point p. Note that Γ'_m is Cartier since it is disjoint from X_{sing}. So

$$\Gamma'_m D \geq \sqrt{\frac{a}{|G_{\text{tors}}|}} \to \infty \text{ as } m \to \infty \tag{3.6}$$

by (3.3). Consequently, there are infinitely many $\mathbb{A}^1$ curves in $X \backslash D$. □

We are also ready to prove Theorem 1.5. First, we need to justify the claim that every $\mathbb{A}^1$ curve in $X_{\mathbb{C}} \backslash D_{\mathbb{C}}$ is defined over $\overline{\mathbb{Q}}$ for a log K3 surface (X, D) over $\overline{\mathbb{Q}}$.

Lemma 3.4. *Let D be an effective divisor of normal crossings on a smooth projective variety X. If $K_X + D$ is pseudo-effective, there do not exist a quasi-projective variety B, a dominant morphism $f : Y = \mathbb{P}^1 \times B \to X$ and a section $\Gamma \subset Y$ of Y/B such that $f^{-1}(D) \subset \Gamma$. In addition, if $\dim X = 2$ and (X, D) is defined over $\overline{\mathbb{Q}}$, then every rational curve $C \subset X_{\mathbb{C}}$ satisfying $|\nu^{-1}(D)| \leq 1$ is defined over $\overline{\mathbb{Q}}$ with $\nu : C^\nu \to X$ the normalization of C.*

Proof. Suppose that such f exists. We may assume that Y is smooth and f is generically finite. Then $f^*(\Omega_X(\log D)) \subset \Omega_Y(\log \Gamma)$. It follows that $(K_Y + \Gamma) - f^*(K_X + D)$ is effective and hence $K_Y + \Gamma$ is pseudo-effective. But $(K_Y + \Gamma).Y_b < 0$ for $b \in B$ general. Contradiction.

Suppose that $\dim X = 2$, (X, D) is defined over $\overline{\mathbb{Q}}$ and $C \subset X_{\mathbb{C}}$ is a rational curve satisfying $|\nu^{-1}(D)| \leq 1$ and transcendental over $\overline{\mathbb{Q}}$. Then C is defined over a field $k \subset \mathbb{C}$ which has finite transcendence degree at least one over $\overline{\mathbb{Q}}$, i.e., C is a rational curve in $X_k = X \times_{\overline{\mathbb{Q}}} k$. We can then find an affine variety B over $\overline{\mathbb{Q}}$ with $\overline{\mathbb{Q}}(B) = k$ such that C can be extended to a non-trivial family $\mathcal{C} \subset X \times_{\overline{\mathbb{Q}}} B$ of rational curves over B whose generic point is $\mathcal{C}_\eta = C$. This construction is usually called *taking a spread over* $\overline{\mathbb{Q}}$. Clearly, $|\nu_b^{-1}(D)| \leq 1$ for $b \in B$ general, where $\nu : \widehat{\mathcal{C}} \to X \times B$ is the normalization of $\mathcal{C}$. So we have a positive dimensional family of $\mathbb{A}^1$ curves in $X \backslash D$. Contradiction. □

Proof of Theorem 1.5. In the proof of Theorem 3.1, we have actually found a sequence $\{\Gamma_n\}$ of rational curves on X such that each Γ_n meets D at a unique point $p_n \in D_{\text{sm}}$ with the properties

$$\begin{aligned} a_n p_n &\in G = i_D^* \operatorname{Pic}(X) \text{ for some } a_n \in \mathbb{Z}^+ \\ m p_n &\notin G \text{ for all } 0 < m < a_n \\ \lim_{n\to\infty} a_n &= \infty. \end{aligned} \tag{3.7}$$

Let M be the subgroup of $\operatorname{Pic}(D)$ generated by p_n. Then (3.7) implies that M contains torsions of arbitrarily high orders.

Suppose that all Γ_n are defined over a number field k. WLOG, let us assume that D is defined over k as well. Then p_n are also defined over k. If D is a smooth elliptic curve, by the Mordell–Weil Theorem (cf. [14]), M is finitely generated and cannot contain torsions of arbitrarily high orders. If D is a nodal rational curve, $M \subset k^*$ again cannot contain torsions of arbitrarily high orders in $(\mathbb{C}^*)_{\text{tors}}$. Contradiction. □

4. Proof of Theorem 1.4

4.1. A necessary condition for the existence of infinitely many $\mathbb{A}^1$ curves

Lemma 4.1. *Let X be a smooth projective surface with $H^1(X) = 0$ and D be a nc divisor on X. If there is an infinite sequence $\{C_m \subset X\}$ of integral curves of increasing degrees satisfying that $|\nu_m^{-1}(D)| \le 1$ for the normalization $\nu_m : C_m^\nu \to X$ of C_m and all m, then*

$$\begin{aligned} &\text{for every log isomorphism } f:\ (X, D) \overset{\sim}{\dashrightarrow} (\widehat{X}, \widehat{D}) \ \text{ satisfying that} \\ &\quad \widehat{X} \text{ is smooth}, \widehat{D} \text{ is of nc and } \mu(\widehat{D}) > 1, \\ &\quad \text{there exist a numerically effective (nef) and big divisor } \widehat{L} \text{ on } \widehat{X} \\ &\quad \text{and irreducible components } \widehat{D}_1 \ne \widehat{D}_2 \text{ of } \widehat{D} \text{ such that} \\ &\qquad \widehat{D}_1 \cap \widehat{D}_2 \ne \emptyset \text{ and } \widehat{L}(\widehat{D} - \widehat{D}_1 - \widehat{D}_2) = 0. \end{aligned} \tag{4.1}$$

Proof. Let $\widehat{C}_m$ be the proper transform of C_m under f. Then $\widehat{C}_m$ meets $\widehat{D}$ at no more than one point. Since $\widehat{D}$ is a nc divisor, no three components of $\widehat{D}$ meet at one point. Therefore, there exist components $\widehat{D}_1 \ne \widehat{D}_2$ of $\widehat{D}$ such that $\widehat{D}_1 \cap \widehat{D}_2 \ne \emptyset$ and $\widehat{C}_m \cap \widehat{D}_i = \emptyset$ for components $\widehat{D}_i \ne \widehat{D}_1, \widehat{D}_2$ and infinitely many m. Hence

$$\widehat{C}_m(\widehat{D} - \widehat{D}_1 - \widehat{D}_2) = 0 \tag{4.2}$$

for infinitely many m. We may simply assume that (4.2) holds for all m.

Next, we claim that there exists a nef and big divisor $\widehat{L} = \sum a_i \widehat{C}_i$ for some $a_i \in \mathbb{Z}$. This, combining with (4.2), will imply $\widehat{L}(\widehat{D} - \widehat{D}_1 - \widehat{D}_2) = 0$.

Obviously, $\widehat{C}_1, \widehat{C}_2, ..., \widehat{C}_m$ are linearly dependent in $H^2(X, \mathbb{Q})$ as long as $m > h^2(X)$. So there exist integers $a_1, a_2, ..., a_m$, not all zero, such that

$$a_1\widehat{C}_1 + a_2\widehat{C}_2 + ... + a_m\widehat{C}_m = 0 \tag{4.3}$$

in $H^2(X, \mathbb{Q})$. We write

$$A = \sum_{a_i>0} a_i\widehat{C}_i \sim_{\text{num}} -\sum_{a_i<0} a_i\widehat{C}_i = B. \tag{4.4}$$

Since $\widehat{C}_i$ are effective, $A \neq 0$ and $B \neq 0$.

Clearly, $AC = BC \geq 0$ for all irreducible curves C and thus A and B are nef. If $A^2 > 0$, A is big and nef and we are done. Suppose that $A^2 = 0$. If $A\widehat{C}_n > 0$ for some $n \in \mathbb{Z}^+$, then $NA + \widehat{C}_n$ is big and nef for some $N >> 1$. Suppose that $A\widehat{C}_n = 0$ for all n.

Since $H^1(X) = 0$, $A = B$ in $\mathrm{Pic}_{\mathbb{Q}}(X)$. WLOG, suppose that $A = B$ in $\mathrm{Pic}(X)$. Then A and B span a base-point-free pencil in $|A|$ which induces a map $f : X \to \mathbb{P}^1$. Since $A\widehat{C}_n = 0$, each $\widehat{C}_n$ is contained in a fiber of f. So $\deg A \geq \deg \widehat{C}_n$ for all n. But $\deg \widehat{C}_n \to \infty$ as $n \to \infty$. Contradiction. Therefore, we must have $A^2 > 0$. Hence A is big and nef and

$$A(\widehat{D} - \widehat{D}_1 - \widehat{D}_2) = 0 \tag{4.5}$$

by (4.2). □

Now we can prove the "only if" part of Theorem 1.4. That is, if there are infinitely many $\mathbb{A}^1$ curves in $X \backslash D$, (X, D) cannot be of type C4.

Proof of "only if" part of Theorem 1.4. If (X, D) is C4, then $D = D_1 + D_2$ with $D_i^2 = \lambda_i > 0$. Let $f : \widehat{X} \to X$ be the blowup of X at the two intersections $D_1 \cap D_2$; then $\widehat{D} = f^{-1}(D)$ has four components, each having self-intersection ≥ -1 (see Fig. 2). We claim that $(\widehat{X}, \widehat{D})$ violates (4.1). Let $\widehat{D}_1 \neq \widehat{D}_2$ be two components of $\widehat{D}$ satisfying $\widehat{D}_1 \cap \widehat{D}_2 \neq \emptyset$. Then

$$\widehat{D} - \widehat{D}_1 - \widehat{D}_2 = \widehat{D}_3 + \widehat{D}_4 \tag{4.6}$$

for two components $\widehat{D}_3 \neq \widehat{D}_4$ satisfying $\widehat{D}_3 \cap \widehat{D}_4 \neq \emptyset$ (see Fig. 2). Since $\widehat{D}_3^2 \geq -1$, $\widehat{D}_4^2 \geq -1$ and $\widehat{D}_3\widehat{D}_4 \geq 1$, $\widehat{D}_3 + \widehat{D}_4$ is nef.

Suppose that (4.1) holds. Then $L(\widehat{D}_3 + \widehat{D}_4) = 0$ for a big and nef divisor L. By the Hodge index theorem [12, Lem. 1.10.2], we have $(\widehat{D}_3 + \widehat{D}_4)^2 = 0$ and $\widehat{D}_3 + \widehat{D}_4$ is numerically trivial. And since X is rational, $\widehat{D}_3 + \widehat{D}_4$ is rationally trivial on X. Contradiction. □

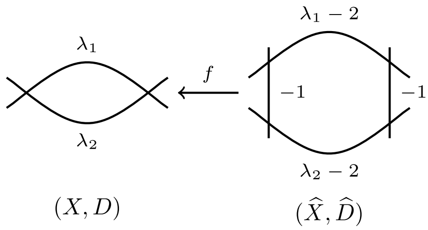

Fig. 2. $D = D_1 + D_2$, $D_1^2 = \lambda_1 > 0$ and $D_2^2 = \lambda_2 > 0$.

4.2. Infinitely many $\mathbb{A}^1$ curves on (X, D) of type C3

It remains to prove the "if" part of Theorem 1.4. That is, there are infinitely many $\mathbb{A}^1$ curves on (X, D) of type C0–C3. We have proved the existence for C0 and C1 in Theorem 3.1. Type C3 is more or less trivial by the following lemma.

Lemma 4.2. *Let (X, D) be a genuine log K3 surface with circular boundary $D = D_1 + D_2$ satisfying that $D_2^2 = 0$. Then there are infinitely many $\mathbb{A}^1$ curves in $X \backslash D$.*

Proof. Let $\pi : X \to \mathbb{P}^1$ be the fiberation given by $|D_2|$. By contracting (-1)-curves contained in the fibers of π, we see that π factors through a rational ruled surface $\overline{X}$ via $f : X \to \overline{X}$. And since

$$-K_{\overline{X}} = f_* D = \overline{D} = \overline{D}_1 + \overline{D}_2 = f_* D_1 + f_* D_2 \tag{4.7}$$

is effective with $\overline{D}_2^2 = 0$, we see that $\overline{X}$ is either $\mathbb{F}_0$ or $\mathbb{F}_1$ by [8, Lem. 7]. Let us replace (X, D) by $(\overline{X}, \overline{D})$.

So $X \cong \mathbb{F}_\beta$, $D_1 \in |2C + (\beta + 1)F|$ and $D_2 \in |F|$, where $\beta = 0$ or 1, F is a fiber of π and C is a section of π with $C^2 = -\beta$. Suppose that D_1 and D_2 meets at two points p and p'. For each $m \in \mathbb{N}$, we have a smooth curve $C_m \in |C + (m + \beta)F|$ meeting D_1 at p with multiplicity $2m + \beta + 1$. Clearly, $C_m \backslash \{p\}$ is an $\mathbb{A}^1$ curve in $X \backslash D$. □

4.3. Infinitely many $\mathbb{A}^1$ curves on (X, D) of type C2

As a technical issue, we need $D_1^2 > 0$ later on. We can assume that by the following lemma.

Lemma 4.3. *Let (X, D) be a genuine log K3 surface with circular boundary $D = D_1 + D_2 + ... + D_n$ satisfying $D_1^2 \neq 0, -1$ and $D_i^2 \leq -2$ for $i \neq 1$. Then there exists a birational morphism $f : X \to \overline{X}$ with $\overline{D} = f_* D$ such that $(\overline{X}, \overline{D})$ is a genuine log K3 surface of either type C1 or type C2 with $\overline{D}_1^2 > 0$ and $\overline{D}_i^2 \leq -2$ for $i \neq 1$.*

Proof. There is nothing to do if $D_1^2 > 0$. Let us assume that $D_i^2 \leq -2$ for all i. We prove by induction on $\text{rank}_{\mathbb{Z}} \text{Pic}(X)$.

Clearly, X is not minimal. Let $f : X \to \overline{X}$ be the contraction of a (−1)-curve followed by a sequence of canonical blowdowns such that $\overline{D}$ has no component with self intersection −1. Clearly, all but one component of $\overline{D}$ satisfy $\overline{D}_i^2 \leq -2$. WLOG, suppose that $\overline{D}_i^2 \leq -2$ for $i \neq 1$. If $\mu(\overline{D}) = 1$ or $\overline{D}_i^2 > 0$, we are done. If $\overline{D}_1^2 \leq -2$, it follows from induction hypothesis since f has reduced $\operatorname{rank}_{\mathbb{Z}} \operatorname{Pic}(X)$ by 1.

Suppose that $\overline{D}_1^2 = 0$. If $\mu(\overline{D}) \geq 3$, assuming $\overline{D}$ of type $(0, \lambda_2, ..., \lambda_m)$, then a sequence of pivot operations at $\overline{D}_1$

$$(0, \lambda_2, ..., \lambda_m) \to (0, \lambda_2 - 1, ..., \lambda_m + 1) \to ... \to (0, -1, ..., \lambda_m + \lambda_2 + 1) \tag{4.8}$$

followed by a sequence of canonical blowdowns will give us what we want.

Suppose that $\overline{D}_1^2 = 0$ and $\mu(\overline{D}) = 2$. If $\overline{D}_1 E = 1$ for some (−1)-curve E, then blow down E and we are done. Otherwise, $\overline{D}_1 E = 0$ for all (−1)-curves E on $\overline{X}$. Then there exists a birational morphism $g : \overline{X} \to \widehat{X}$ with $\widehat{D}_i = g_* \overline{D}_i$ such that $\widehat{D}_2^2 = -1$. Blowing down $\widehat{D}_2$, we obtain a genuine log K3 of type C1. □

It remains to prove the following:

Proposition 4.4. *Let (X, D) be a genuine log K3 surface with circular boundary $D = D_1 + D_2 + ... + D_n$ of type $(\lambda_1, \lambda_2, ..., \lambda_n)$. If $\lambda_1 > 0$ and $\lambda_i \leq -2$ for $i \neq 1$, then there are infinitely many $\mathbb{A}^1$ curves in $X \backslash D$.*

We follow a similar line argument as Xu's proof of Theorem 3.2. To start with, we need "many" rational curves in X disjoint from $D_2 + D_3 + ... + D_n$, or equivalently, many rational curves in the smooth locus $\overline{X}_{\mathrm{sm}}$ of $\overline{X}$, where $X \to \overline{X}$ is the contraction of $D_2 + D_3 + ... + D_n$. This is where the theorem of Keel and McKernan comes in: the smooth locus of a log del Pezzo surface is rationally connected [9, Corollary 1.6]. We put their theorem in the following form:

Theorem 4.5 *(Keel–McKernan). Let X be a log del Pezzo surface. Then there exists an ample Cartier divisor A on X such that*

- *$V_{A,0}$ is nonempty of expected dimension $-K_X A - 1 \geq 2$,*
- *a general member $C \in V_{A,0}$ lies inside X_{sm},*
- *the normalization $\nu : C^\nu \to X$ of C is an immersion,*
- *$\nu^* T_X$ is ample,*
- *and C meets a fixed reduced curve D transversely.*

Furthermore, the same holds for $V_{mA,0}$ and all $m \in \mathbb{Z}^+$ and a general member of $V_{mA,0}$ is nodal if $-mK_X A - 1 \geq 4$.

Remark 4.6. The key observation here is that once we can deform a rational curve away from X_{sing}, the standard deformation theory of curves on smooth surfaces will take over.

As long as $C \cap X_{\text{sing}} = \emptyset$ for a general member $C \in V_{A,0}$, we can prove that $V_{A,0}$ has the expected dimension. In addition, as long as $\dim V_{A,0} \geq 2$, a general member $C \in V_{A,0}$ behaves as expected (cf. [7, Chapter 3, Section B]), i.e., $\nu : C^\nu \to X$ is an immersion, $\nu^* T_X$ is ample and C is nodal if $\dim V_{A,0} \geq 4$. In our case, to deform a rational curve away from the only singularity of X' or $\overline{X}$, we actually only need a lemma in Keel–McKernan's paper [9, Lemma 6.4]. Moreover, once we have $\dim V_{A,0} = -K_X A - 1$, we can produce more rational curves by taking two general members $C_1, C_2 \in V_{A,0}$ and deforming the union $C_1 \cup C_2$ to a rational curve in $V_{2A,0}$. More generally, if $\dim V_{A_1,0} = -K_X A_1 - 1 \geq 0$, $\dim V_{A_2,0} = -K_X A_2 - 1 \geq 0$ and two general members $C_1 \in V_{A_1,0}$ and $C_2 \in V_{A_2,0}$ meet transversely, then $C_1 \cup C_2$ can be deformed to a rational curve in $V_{A_1+A_2,0}$. So we can prove in this way that the theorem holds for all $V_{mA,0}$.

Basically, we want to impose tangency conditions on $C \in V_{A,0}$. Let us first define the subvarieties of Severi varieties of curves on X tangent to a fixed curve D as follows.

Definition 4.7. For a curve D on a projective surface X and a zero cycle $\alpha = m_1 p_1 + m_2 p_2 + ... + m_k p_k \in Z_0(D)$, we use the notation $V_{A,g,D,\alpha}$ to denote the subvariety of $V_{A,g}$ consisting of integral curves $C \in |A|$ of genus g satisfying that

- C meets D properly and
- there exists $q_i \in \nu^{-1}(p_i)$ and $n_i \geq m_i$ such that $q_1, q_2, ..., q_k$ are distinct and $\nu^* D = n_i q_i$ when ν is restricted to the open neighborhoods of p_i and q_i for $i = 1, 2, ..., k$,

where $\nu : \widehat{C} \to X$ is the normalization of C, $m_1, m_2, ..., m_k \in \mathbb{Q}^+$ and $p_1, p_2, ..., p_k$ are points on D such that D is locally $\mathbb{Q}$-Cartier at each p_i.

We are going to prove Proposition 4.4 by showing that there are infinitely many rational curves $C \subset X$ meeting D only at $p \in D_1 \cap D_2$; more precisely, we are going to show

$$V_{A_m,0,D,a_m p} \neq \emptyset \tag{4.9}$$

for a sequence of divisors A_m satisfying $a_m = A_m D \to \infty$ as $m \to \infty$. For starters, we prove the following:

Proposition 4.8. *Let (X, D) be a genuine log K3 surface with circular boundary $D = D_1 + D_2 + ... + D_n$ of type $(\lambda_1, \lambda_2, ..., \lambda_n)$. If $\lambda_1 > 0$ and $\lambda_i \leq -2$ for $i \neq 1$, then (X, D) can be replaced by a log isomorphic model such that $D_1^2 > 0$, the intersection matrix of $D - D_1$ is negative definite and there exists a sequence of divisors A_m on X satisfying that*

$$\begin{aligned} A_m(D - D_1) &= A_m D_2 = 1, \\ \lim_{m\to\infty} A_m D &= \infty, \\ \dim V_{A_m,0,D,2p} &= A_m D - 2 \textit{ for } p \in D_1 \cap D_2, \\ \textit{and a general member } \; C_m &\in V_{A_m,0,D,2p} \textit{ meets } D \\ &\textit{transversely at } A_m D - 2 \textit{ points outside of } p. \end{aligned} \tag{4.10}$$

Proof. Let us first prove that (X, D) can be replaced by a log isomorphic model such that $D_1^2 > 0$, the intersection matrix of $D - D_1$ is negative definite and there is an effective divisor F on X such that $F^2 = 0$ and $FD_1 = FD_2 = 1$.

By (2.1), we have that $\operatorname{rank}_{\mathbb{Z}} \operatorname{Pic}(X) \geq n$ for the log K3 pair (X, D). If $n = 2$, then there exists a fiberation $\pi : X \to \mathbb{P}^1$ whose general fibers are $\mathbb{P}^1$ and a fiber F of π has the required property. Suppose that $n \geq 3$. Then X is a rational surface of $\operatorname{rank}_{\mathbb{Z}} \operatorname{Pic}(X) \geq 3$. In particular, X contains a (-1)-curve E. Since $DE = -K_X E = 1$, $D_k E = 1$ for some $k \neq 1$. We prove that there exists a log isomorphism $f : (X, D) \dashrightarrow (\widehat{X}, \widehat{D})$ such that the proper transform of E is the divisor F we want. This f is given by a sequence of canonical blowups and blowdowns and pivot operations. First, we can replace D_1 by a chain of curves of self intersections $(-2, -2, ..., -2, -1, 0)$ by a sequence of canonical blowups over $D_1 \cap D_n$:

$$\begin{aligned} &(\lambda_1, \lambda_2, ..., \lambda_n) \\ &\to (\lambda_1 - 1, \lambda_2, ..., \lambda_n - 1, -1) \\ &\to (\lambda_1 - 2, \lambda_2, ..., \lambda_n - 1, -2, -1) \to ... \\ &\to (0, \lambda_2, \lambda_3, ..., \lambda_n - 1, -2, -2, ..., -2, -1). \end{aligned} \tag{4.11}$$

Then a sequence of pivots at D_1 render $D_2^2 = -1$, D_2 can then be contracted and D_1^2 is restored to 0 by a canonical blowup:

$$\begin{aligned} &\to (0, -1, \lambda_3, ..., \lambda_n - 1, -2, -2, ..., -2, \lambda_2) \\ &\to (1, \lambda_3 + 1, \lambda_4, ..., \lambda_n - 1, -2, -2, ..., -2, \lambda_2) \\ &\to (0, \lambda_3 + 1, \lambda_4, ..., \lambda_n - 1, -2, -2, ..., -2, \lambda_2 - 1, -1). \end{aligned} \tag{4.12}$$

We continue this process until D_k is contracted:

$$\begin{aligned} &\to (0, -1, \lambda_4, ..., \lambda_n - 1, -2, -2, ..., -2, \lambda_2 - 1, \lambda_3 + 1) \\ &\to (1, \lambda_4 + 1, ..., \lambda_n - 1, -2, -2, ..., -2, \lambda_2 - 1, \lambda_3 + 1) \\ &\to (0, \lambda_4 + 1, ..., \lambda_n - 1, -2, -2, ..., -2, \lambda_2 - 1, \lambda_3, -1) \\ &\to (0, -1, \lambda_5, ..., \lambda_n - 1, -2, ..., -2, \lambda_2 - 1, \lambda_3, \lambda_4 + 1) \\ &\to (1, \lambda_5 + 1, ..., \lambda_n - 1, -2, ..., -2, \lambda_2 - 1, \lambda_3, \lambda_4 + 1) \to ... \end{aligned}$$

$$\to (1, \lambda_k + 1, ..., \lambda_n - 1, -2, ..., -2, \lambda_2 - 1, \lambda_3, ..., \lambda_{k-2}, \lambda_{k-1} + 1) \tag{4.13}$$
$$\to (0, \lambda_k + 1, ..., \lambda_n - 1, -2, ..., -2, \lambda_2 - 1, \lambda_3, ..., \lambda_{k-2}, \lambda_{k-1}, -1)$$
$$\to (0, -1, \lambda_{k+1}, ..., \lambda_n - 1, -2, ..., -2, \lambda_2 - 1, \lambda_3, ..., \lambda_{k-1}, \lambda_k + 1)$$
$$\to (1, \lambda_{k+1} + 1, ..., \lambda_n - 1, -2, ..., -2, \lambda_2 - 1, \lambda_3, ..., \lambda_{k-1}, \lambda_k + 1)$$

where (4.11)–(4.13) illustrate how the type of circular boundary D changes in the process. At the last step, when we contract the proper transform $\widehat{D}_k$ of D_k, the self-intersection E^2 of E increases by 1 and its proper transform F is what we are after.

Applying Theorem 4.5 to the log del Pezzo surfaces $\overline{X}$ obtained from X by contracting $D - D_1$, we obtain base-point-free (bpf) and big divisors A on X such that

$$\begin{aligned} A(D - D_1) &= 0, \\ \dim V_{A,0} &= AD - 1 \geq 2. \end{aligned} \tag{4.14}$$

Let $A_m = mA + F$. Clearly, $A_m D_2 = 1$, $A_m D_i = 0$ for $i \neq 1, 2$ and $A_m D \to \infty$ as $m \to \infty$.

Let F_p be the member of the pencil $|F|$ passing through p. Then the union $\Gamma_1 \cup \Gamma_2 \cup ... \cup \Gamma_m \cup F_p$ can be deformed to a curve $C_m \in V_{A_m,0,D,2p}$ for m general members $\Gamma_i \in V_{A,0}$. Here we only need to apply the standard deformation theory (cf. [7]) to curves in $|A_m|$ passing through p (also see Remark 4.6). So

$$\dim V_{A_m,0,D,2p} = A_m D - 2. \tag{4.15}$$

Using the standard deformation theory and the rigidity Lemma 3.4, it is easy to see that a general member $C_m \in V_{A_m,0,D,2p}$ meets D transversely at $A_m D - 2$ points outside of p. □

Starting with (4.15), naturally, we try to prove (4.9) by imposing more tangency conditions on $C_m \in V_{A_m,0,D,2p}$ at p. We are going to do this inductively by increasing the multiplicity at p one at a time. That is, we will roughly show that

$$\dim V_{A_m,0,D,kp} = A_m D - k \tag{4.16}$$

for all k. When we deform/degenerate a family of rational curves on X for this purpose, one difficulty arises: its flat limit might contain some components of D. To deal with this situation, we need the following key lemma.

Lemma 4.9. *Let X be a smooth projective surface, $D = D_1 + D_2 + ... + D_n$ be a circular boundary on X and $f : Y/\Delta \to X$ be a family of stable rational maps over the unit disk $\Delta = \{|t| < 1\}$ satisfying that $Y_t \cong \mathbb{P}^1$ and $f(Y_t)$ meets D properly for $t \neq 0$. Suppose that*

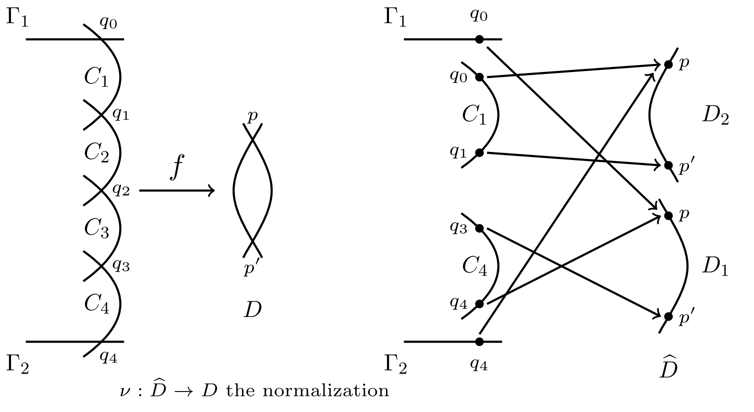

Fig. 3. A configuration of G° for $f : G^\circ \to D = D_1 + D_2$.

$$\begin{gathered} \dim(D \cap f(Y_0)) > 0, \\ f^*D = \sum_{i=1}^{c} m_i \Gamma_i + V \textit{ and} \\ D_{\text{sing}} \cap f(\Gamma_i) = \emptyset \textit{ for } i \neq 1, 2, \end{gathered} \tag{4.17}$$

where $V \subset Y_0$ *and* Γ_i *are distinct sections of* Y/Δ *of multiplicities* $m_i > 0$ *in* f^*D.

Let $G = \Gamma_1 \cup \Gamma_2 \cup \operatorname{supp}(V) \subset Y$ *and let* G° *be the curve obtained from* G *by contracting all contractible components under the map* $f : G \to D$. *That is,* $G \to G^\circ \to D$ *is the Stein factorization of* $f : G \to D$. *Then* G° *is a chain of curves given by (see Fig. 3 for a configuration of* G° *when* $n = 2$*)*

$$G^\circ = \Gamma_1 \cup C_1 \cup C_2 \cup \ldots \cup C_a \cup \Gamma_2 \tag{4.18}$$

where $\Gamma_1 \cap C_1 = q_0$, $C_i \cap C_{i+1} = q_i$, $C_a \cap \Gamma_2 = q_a$,

- $f(q_i) \in D_{sing}$ *for* $i = 0, 1, ..., a$,
- f *sends each* C_i *onto one of* D_j *with a map totally ramified over the two intersections* $D_j \cap (D - D_j)$ *for* $i = 1, 2, ..., a$,
- f *maps* G° *locally at* q_i *surjectively onto* D *at* $f(q_i)$ *for* $0 < i < a$,
- f *maps* G° *locally at* q_0 *surjectively onto* D *at* $f(q_0)$ *if* $f_*\Gamma_1 \neq 0$,
- *and* f *maps* G° *locally at* q_a *surjectively onto* D *at* $f(q_a)$ *if* $f_*\Gamma_2 \neq 0$.

In particular, if $f_*\Gamma_1 \neq 0$ *and* $f(\Gamma_1) \subset D_1$, Γ_1 *lies on the connected component* M *of* $f^{-1}(D_1)$ *such that* $\Gamma_i \not\subset M$ *for all* $i \neq 1$ *and* $f_*E = 0$ *for all irreducible components* $E \neq \Gamma_1 \subset M$.

Remark 4.10. Here is some clarification of the terms in the lemma:

- We call a curve $F = F_1 \cup F_2 \cup \ldots \cup F_n$ a *chain* of curves if the dual graph of F is a chain, i.e., a tree with at most two vertices of degree ≤ 1.
- The curve G° has a node at each q_i and f maps each q_i to a node of D. By "f maps G° locally at q_i surjectively onto D at $f(q_i)$", we mean f maps the two branches of G° at q_i to the two branches of D at $f(q_i)$. More explicitly, f is locally given by $f(x, y) = (x^m, y^n)$ from $\operatorname{Spec} \mathbb{C}[[x, y]]/(xy)$ to $\operatorname{Spec} \mathbb{C}[[s, t]]/(st)$ for some positive integers m and n.

Proof of Lemma 4.9. We first prove the following statement:

Claim 4.11. *For every component $C \subset Y_0$ and a point $q \in C$ satisfying that $f_* C \neq 0$, $f(C) \subset D$ and $f(q) \in D_{sing}$, there exists either a chain $C \cup E_1 \cup E_2 \cup \ldots \cup E_a \cup \Gamma_i$ of curves satisfying*

$$\begin{gathered} C \cap E_1 = q, E_k \cap E_{k+1} \neq \emptyset, f_* E_k = 0, E_a \cap \Gamma_i = q' \text{ for some } \Gamma_i \\ \text{and } f \text{ maps } C \text{ and } \Gamma_i \text{ locally at } q \text{ and } q' \\ \text{to the two branches of } D \text{ at } f(q), \text{ respectively, if } f_* \Gamma_i \neq 0 \end{gathered} \tag{4.19}$$

or a chain $C \cup E_1 \cup E_2 \cup \ldots \cup E_a \cup C'$ of curves satisfying

$$\begin{gathered} C \cap E_1 = q, E_k \cap E_{k+1} \neq \emptyset, f_* E_k = 0, E_a \cap C' = q' \\ \text{for some component } C' \subset Y_0 \text{ with } f_*(C') \neq 0 \text{ and } f(C') \subset D \\ \text{and } f \text{ maps } C \text{ and } C' \text{ locally at } q \text{ and } q' \\ \text{to the two branches of } D \text{ at } f(q), \text{ respectively.} \end{gathered} \tag{4.20}$$

WLOG, we assume that $f(C) = D_1$ and $f(q) = p = D_1 \cap D_2$. The statement is local. So we choose an analytic open neighborhood U of $p \in X$ and let M be the connected component of $f^{-1}(U)$ that contains q. Since $q \in f^{-1}(D_2)$, we have either $\Gamma_i \cap M \neq \emptyset$ for some Γ_i with $f(\Gamma_i \cap M) \subset D_2$ or $C' \cap M \neq \emptyset$ for some component $C' \subset Y_0$ with $f(C' \cap M) = D_2 \cap U$ such that Γ_i or C' is joined to C by a chain of contractible components. In addition, if it is the former and $f_* \Gamma_i \neq 0$, we necessarily have $f(\Gamma_i \cap M) = D_2 \cap U$. This proves Claim 4.11.

We let Σ be the subgraph of the dual graph of Y_0 that contains all components $C \subset Y_0$ satisfying $f_* C \neq 0$ and $f(C) \subset D$ and all chains $C \cup E_1 \cup E_2 \cup \ldots \cup E_a \cup C'$ with the property (4.20). Note that every contractible component in Σ has degree ≥ 2.

If $D_{\text{sing}} \cap f(\Gamma_i) = \emptyset$ for all but one Γ_i, then by Claim 4.11, all but one vertices in Σ have degree ≥ 2 with the remaining vertex of degree ≥ 1, which contradicts the fact that Σ is a disjoint union of trees.

Therefore, $D_{\text{sing}} \cap f(\Gamma_i) \neq \emptyset$ for $i = 1, 2$ and all but two vertices in Σ have degree ≥ 2 and the remaining two vertices have total degree ≥ 2. So Σ has to be a chain. Then it is easy to see that G° is a chain of curves with the properties described by the lemma. □

To finish the proof of Proposition 4.4 and thus settle the last case of Theorem 1.4, it remains to prove the following:

Proposition 4.12. *Let X be a smooth projective surface, $D = \sum_{i=1}^n D_i$ be a circular boundary on X, $p \in D_1 \cap D_2$ and A be a divisor on X satisfying that $a = AD = A(D_1 + D_2) > AD_2 = 1$. If $K_X + D$ is pseudo-effective, $D_1^2 > 0$, $\dim V_{A,0,D,2p} = a - 2$ and a general member of $V_{A,0,D,2p}$ meets D transversely at $a-2$ points outside of p, then for each $0 \leq l \leq a-2$, there exist $m_0, m_1, ..., m_{a-l-2} \in \mathbb{Z}^+$ and $A_l \leq A$ such that*

$$\begin{aligned} & a = A_l D = 1 + \sum_{i=0}^{a-l-2} m_i \text{ and} \\ & V_{A_l,0,D,\alpha} \neq \emptyset \text{ for } \alpha = (m_0+1)p + \sum_{i=1}^{a-l-2} m_i p_i \end{aligned} \tag{4.21}$$

where $p_1, p_2, ..., p_{a-l-2}$ are $a-l-2$ general points on D_1 and we write $A \geq B$ for two divisors A and B if $A - B$ is effective.

Proof. Since a general member of $V_{A,0,D,2p}$ meets D transversely at $a-2$ points outside of p, we have

$$V_{A,0,D,\alpha} \neq \emptyset \text{ for } \alpha = 2p + p_1 + p_2 + ... + p_{a-2} \tag{4.22}$$

where $p_1, p_2, ..., p_{a-2}$ are $a-2$ general points on D_1. So the proposition holds for $l = 0$. We argue by induction on l.

Suppose that there exist $m_0, m_1, ..., m_{a-l-2} \in \mathbb{Z}^+$ such that

$$\dim V_{A,0,D,\alpha} = 0 \text{ for } \alpha = (m_0+1)p + \sum_{i=1}^{a-l-2} m_i p_i. \tag{4.23}$$

Among p_i, we fix $a-l-3$ general points $p_2, p_3, ..., p_{a-l-2}$ on D_1, let

$$\lambda = \sum_{i=2}^{a-l-2} m_i p_i \tag{4.24}$$

and let $q = p_1$ vary. More precisely, we consider the closure W of

$$\begin{aligned} W^\circ = \{(C, q) : C \in V_{A,0,D,(m_0+1)p+m_1 q+\lambda} \text{ and } q \in D_1 \text{ general}\} \\ \subset |A| \times D_1. \end{aligned} \tag{4.25}$$

By induction hypothesis (4.23), W is finite over D_1.

Let $U = \{(C, q, s) : s \in C, (C, q) \in W\} \subset W \times X$ be the universal family over W. Applying stable reduction to U/D_1, we obtain a finite morphism $\phi : B \to W \to D_1$ and a family $f : Y/B \to X$ of stable rational maps satisfying that $f_* Y_b \subset U_{\phi(b)}$ for all $b \in B$.

Obviously,

$$f^* D_1 = m_0 P + m_1 Q + \sum_{i=2}^{a-l-2} m_i P_i + V = m_0 P + m_1 Q + \Lambda + V \tag{4.26}$$

where $\pi_* V = 0$ for $\pi : Y \to B$ and P, P_i and Q are the sections of Y/B satisfying that $f(P) = p$, $f(P_i) = p_i$ and $f(Q \cap Y_b) = \phi(b)$ for all $b \in B$.

In other words, Q is the moving intersections between $f_* Y_b$ and D_1, while P and P_i are the fixed intersections. We want to show that Q "collides" with one of P and P_i, which will reduce the number of points in $\{p_1, p_2, ..., p_{a-l-2}\}$ and thus increase l by one.

One of the key hypotheses is $D_1^2 > 0$. So D_1 is nef and big. Consequently, $f^{-1}(D_1)$ is connected. So Q and one of P and P_i are joined by a chain of curves in V. More precisely, either $P + V_0 + Q$ or $P_i + V_0 + Q$ is connected for some i and a connected component V_0 of V contained in a fiber Y_b. We will be almost done if $f(Y_b)$ meets D properly, i.e.,

$$\dim(f(Y_b) \cap D) = 0, \tag{4.27}$$

which implies that $f_* V_0 = 0$. This is guaranteed by our key lemma. Suppose that $f(Y_b)$ fails to meet D properly. Here we only consider Y/B in an analytic open neighborhood of Y_b. Suppose that Q lies on a connected component M of $f^{-1}(D_1)$. Applying Lemma 4.9 to Y/B with $\Gamma_1 = Q$ and $\Gamma_2 = P$, we conclude that M does not contain any sections $\Gamma \neq Q$ of Y/B with $f(\Gamma) \subset D_1$. That is, $P, P_i \not\subset M$ for all i. But M must contain one of $P + V_0 + Q$ and $P_i + V_0 + Q$ since one of $P + V_0 + Q$ and $P_i + V_0 + Q$ is connected. Contradiction. So we necessarily have (4.27).

Therefore, $C = f_* Y_b$ meets D properly and $f_* V_0 = 0$. Hence

$$C.D = (m_0 + m_1 + 1)p + \sum_{j=2}^{a-l-2} m_j p_j \tag{4.28}$$

if $P + V_0 + Q$ is connected and

$$C.D = (m_0 + 1)p + (m_1 + m_i)p_i + \sum_{\substack{2 \le j \le a-l-2 \\ j \neq i}} m_j p_j \tag{4.29}$$

if $P_i + V_0 + Q$ is connected for some $2 \le i \le a - l - 2$.

We claim that we cannot write $C = C_1 + C_2$ with $C_k \ge 0$ and $C_k D > 0$. Otherwise, since $CD_2 = 1$, one of C_1 and C_2 does not pass through p. Thus,

$$C_1.D = a_{11}p + \sum_{j=2}^{a-l-2} a_{1j}p_j$$
$$C_2.D = \sum_{j=2}^{a-l-2} a_{2j}p_j \tag{4.30}$$

for some $a_{1j}, a_{2j} \in \mathbb{N}$. So C_2 meets D at the smooth points on D_1 and hence

$$\sum_{j=2}^{a-l-2} a_{2j}p_j = i_D^* C_2 \in i_D^* \operatorname{Pic}(X) \tag{4.31}$$

in $\operatorname{Pic}(D)$. Note that $i_D^* \operatorname{Pic}(X)$ is a finitely generated subgroup of $\operatorname{Pic}(D)$. On the other hand, $p_2, p_3, ..., p_{a-l-2}$ are general points on D_1. So (4.31) cannot hold.

This implies that C has only one irreducible component Γ that meets D and the union of the rest of the components of C, called E, are disjoint from D. So we have the decomposition

$$C = \Gamma + E \tag{4.32}$$

where $\Gamma D = AD = a$, which is equivalent to saying that $ED = 0$. Now we have that $\Gamma \in V_{A_{l+1},0,D,\alpha}$ for $A_{l+1} = \Gamma$ and

$$\alpha = (m_0 + m_1 + 1)p + \sum_{j=2}^{a-l-2} m_j p_j \text{ or}$$
$$\alpha = (m_0 + 1)p + (m_1 + m_i)p_i + \sum_{\substack{2 \le j \le a-l-2 \\ j \ne i}} m_j p_j \tag{4.33}$$

for some $2 \le i \le a - l - 2$. □

5. Iitaka models

5.1. Iitaka models

Iitaka had a complete classification of log K3's. More generally, Iitaka and Zhang also classified Iitaka surfaces, which are log K3's with the condition $h^0(\Omega_X(\log D)) = 0$ removed. For our purpose, we just need the following [8, Theorem 3 & Theorem $\mathrm{II_a}$ & Table $\mathrm{II_a}$ & Proposition 16 & Table $\mathrm{II_b}$]

Theorem 5.1 *(Iitaka's Classifications of Type II log K3). For every Type* II *log K3* (X, D)*, there exists a birational morphism* $f : X \to \overline{X}$*, where* $\overline{X}$ *is a minimal rational surface,* $\overline{D} = f_*D$ *is a nc divisor and* $(\overline{X}, \overline{D})$ *is one of the following:*

(a-i) $(\mathbb{P}^2, E)$ *where* E *is a smooth elliptic curve;*
(a-ii) $(\mathbb{F}_0, E)$ *where* E *is a smooth elliptic curve;*
(a-iii) $(\mathbb{F}_2, E)$ *or* $(\mathbb{F}_2, E + \Delta_\infty)$*, where* E *is a smooth elliptic curve and* Δ_∞ *is the section of* $\mathbb{F}_2/\mathbb{P}^1$ *with* $\Delta_\infty^2 = -2$*;*
(b-i) $(\mathbb{P}^2, H_1 + H_2 + H_3)$ *where each* H_i *is a line on* $\mathbb{P}^2$*;*
(b-ii) $(\mathbb{F}_0, H_1 + H_2 + G_1 + G_2)$ *where each* H_i *has type* $(1, 0)$ *and each* G_j *has type* $(0, 1)$*;*
(b-iii) $(\mathbb{F}_\beta, \Delta_\lambda + \Delta_\infty + F_1 + F_2)$ *where* Δ_λ *and* Δ_∞ *are two sections of* $\mathbb{F}_\beta/\mathbb{P}^1$ *satisfying* $\Delta_\lambda^2 = -\Delta_\infty^2 = \beta \geq 2$ *and each* F_i *is a fiber;*
(b-iv) $(\mathbb{P}^2, H + C)$ *where* H *is a line and* C *is a conic;*
(b-v) $(\mathbb{F}_0, C_1 + C_2)$ *where each* C_i *has type* $(1, 1)$*;*
(b-vi) $(\mathbb{F}_2, \Delta_0 + \Delta_\lambda)$ *or* $(\mathbb{F}_2, \Delta_0 + \Delta_\lambda + \Delta_\infty)$ *where* Δ_0*,* Δ_λ *and* Δ_∞ *are sections of* $\mathbb{F}^2/\mathbb{P}^1$ *satisfying* $\Delta_0^2 = \Delta_\lambda^2 = -\Delta_\infty^2 = 2$*;*
(b-vii) $(\mathbb{F}_\beta, F + \Delta_\infty + C_3)$ *where* F *is a fiber and* Δ_∞ *and* C_3 *are two sections of* $\mathbb{F}_\beta/\mathbb{P}^1$ *satisfying* $-\Delta_\infty^2 = C_3^2 - 2 = \beta \geq 2$*;*
(b-viii) $(\mathbb{F}_0, H + G + C)$ *where* H, G *and* C *has types* $(1, 0)$*,* $(0, 1)$ *and* $(1, 1)$*, respectively;*
(b-ix) $(\mathbb{P}^2, E)$ *where* E *is a nodal rational curve with one node;*
(b-x) $(\mathbb{F}_0, E)$ *where* E *is a nodal rational curve with one node;*
(b-xi) $(\mathbb{F}_2, E)$ *or* $(\mathbb{F}_2, E + \Delta_\infty)$*, where* E *is a nodal rational curve with one node and* Δ_∞ *is the section of* $\mathbb{F}_2/\mathbb{P}^1$ *with* $\Delta_\infty^2 = -2$*;*
(b-xii) $(\mathbb{F}_0, C_1 + C_2)$ *where* C_1 *has type* $(1, 2)$ *and* C_2 *has type* $(1, 0)$*;*
(b-xiii) $(\mathbb{F}_\beta, C + \Delta_\infty)$ *where* C *and* Δ_∞ *are two sections of* $\mathbb{F}_\beta/\mathbb{P}^1$ *satisfying* $-\Delta_\infty^2 = C^2 - 4 = \beta \geq 2$*.*

We use the notations $\text{II}_{\text{a-}\bullet}$ and $\text{II}_{\text{b-}\bullet}$ to refer such $(\overline{X}, \overline{D})$. For the last type $\text{II}_{\text{b-xiii}}$, we may contract Δ_∞ to obtain a log del Pezzo surface. Thus, we replace/expand this type by/to the following:

(b-xiii) $\overline{X}$ is a log del Pezzo surface of Picard rank 1, i.e., $\overline{X}$ is a projective surface with log terminal singularities, ample anti-canonical divisor $-K_{\overline{X}}$ and $\operatorname{rank}_{\mathbb{Z}} \operatorname{Pic}(\overline{X}) = 1$, $\overline{D} \sim -K_{\overline{X}}$ is a rational curve with one node $\overline{p}$ and $\overline{X}$ is singular at $\overline{p}$ and smooth outside of $\overline{p}$.

Log del Pezzo surfaces of Picard rank 1 have been extensively studied (cf. [9]). In our case, log del Pezzo surfaces of Picard rank 1 with a unique singularity were classified by H. Kojima [10]. Although we do not need it here, one can use Kojima's classification to further divide $\text{II}_{\text{b-xiii}}$ into subclasses.

We call $(\overline{X}, \overline{D})$ an *Iitaka model* of (X, D). Note that Iitaka model for a log K3 is not unique. For example, let X be the blowup of $\mathbb{P}^2$ at two distinct points and let $D = C_1 + C_2$, where $C_1 \sim 2H$ and $C_2 \sim H - E_1 - E_2$ with H the pullback of the hyperplane divisor and E_i the exceptional divisors of $X \to \mathbb{P}^2$. We may let $f : X \to \overline{X} \cong \mathbb{P}^2$ be the blowdown of E_1 and E_2, which results in Iitaka model $\text{II}_{\text{b-iv}}$. Or we may let $f : X \to \overline{X} \cong \mathbb{F}_0$ be the blowdown of C_2, which results in Iitaka model $\mathrm{II}_{\text{b-x}}$. Indeed, although we do not need this fact, it is easy to show that there exists a genuine log K3 whose Iitaka model can be any type in II_{b}.

Also note that although we have $K_X + D = f^*(K_{\overline{X}} + \overline{D})$, $(\overline{X}, \overline{D})$ is not necessarily a log K3. That is, $(\overline{X}, \overline{D})$ might be irregular:

$$h^0(\Omega_{\overline{X}}(\log \overline{D})) = \dim_{\mathbb{Q}} \ker(\oplus \mathbb{Q}\overline{D}_i \to H^2(\overline{X}, \mathbb{Q})) > 0 \tag{5.1}$$

for Iitaka types $\mathrm{II}_{\text{b-i}}$–$\mathrm{II}_{\text{b-viii}}$, where $\overline{D}_i$ are the irreducible components of $\overline{D}$.

We can reformulate our theorems using the language of Iitaka model: there are infinitely many $\mathbb{A}^1$ curves in $X\backslash D$ if and only if (X, D) has a log K3 Iitaka model, i.e., $\mathrm{II}_{\text{b-ix}}$–$\mathrm{II}_{\text{b-xiii}}$.

Theorem 5.2. *For every genuine log K3 surface (X, D) of type* II*, there exists a log isomorphism $(X, D) \dashrightarrow^{\sim} (\widehat{X}, \widehat{D})$ followed by a birational morphism $f : \widehat{X} \to \overline{X}$ with $\overline{D} = f_*\widehat{D}$ such that $(\widehat{X}, \widehat{D})$ is one of C0–C4 in Theorem 1.3 and*

- *if $(\widehat{X}, \widehat{D})$ is C0, $(\overline{X}, \overline{D})$ is one of* $\mathrm{II}_{\text{a-}\bullet}$*;*
- *if $(\widehat{X}, \widehat{D})$ is C1, $(\overline{X}, \overline{D})$ is one of* $\mathrm{II}_{\text{b-ix}}$–$\mathrm{II}_{\text{b-xi}}$*;*
- *if $(\widehat{X}, \widehat{D})$ is C2, $(\overline{X}, \overline{D})$ is one of* $\mathrm{II}_{\text{b-ix}}$–$\mathrm{II}_{\text{b-xiii}}$*;*
- *if $(\widehat{X}, \widehat{D})$ is C3, $(\overline{X}, \overline{D})$ is* $\mathrm{II}_{\text{b-xii}}$*;*
- *if $(\widehat{X}, \widehat{D})$ is C4, $(\overline{X}, \overline{D})$ is* $\mathrm{II}_{\text{b-iv}}$–$\mathrm{II}_{\text{b-vi}}$*.*

On the other hand, we can prove

Theorem 5.3. *For each $(\overline{X}, \overline{D})$ among* $\mathrm{II}_{\text{b-i}}$–$\mathrm{II}_{\text{b-viii}}$*, there exists a genuine log K3 surface (X, D) and a birational morphism $g : X \to \overline{X}$ with $\overline{D} = g_*D$ such that there are at most finitely many $\mathbb{A}^1$ curves in $X\backslash D$.*

The proof of Theorem 5.3 gives many examples of genuine log K3 surfaces without infinitely many $\mathbb{A}^1$ curves.

5.2. Proof of Theorem 5.2

If $(\widehat{X}, \widehat{D})$ is of type C0, C1 or C4, we simply let f be the blowdown of X to a minimal rational surface $\overline{X}$.

Suppose that $(\widehat{X}, \widehat{D})$ is of type C3. Let $\pi : \widehat{X} \to \mathbb{P}^1$ be the fiberation given by $|\widehat{D}_2|$. There exists a sequence of blowdowns of (-1)-curves contained in the fibers of π:

$$\widehat{X} = \widehat{X}_n \xrightarrow{f_n} \widehat{X}_{n-1} \xrightarrow{f_{n-1}} \ldots \xrightarrow{f_2} \widehat{X}_1 \xrightarrow{f_1} \widehat{X}_0 = \overline{X} \tag{5.2}$$

such that π factors through $f = f_1 \circ f_2 \circ \ldots \circ f_n$ and $\overline{X}$ is a rational ruled surface with $\overline{\pi} = \pi \circ f^{-1} : \overline{X} \to \mathbb{P}^1$. Since $\overline{D} = \overline{D}_1 + \overline{D}_2$ with $\overline{D}_2^2 = 0$, we see that $\overline{X}$ must be either $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_1$. If it is the former, we are done.

Suppose that $\overline{X} \cong \mathbb{F}_1$. If $\widehat{X} \ncong \mathbb{F}_1$, then $n \geq 1$ in (5.2) and let E_1 be the exceptional curve of f_1; clearly, there exists another (-1)-curve E_2 such that $E_1 + E_2$ is a fiber of $\overline{\pi} \circ f_1 : \widehat{X}_1 \to \mathbb{P}^1$ and the blowdown $f_1' : \widehat{X}_1 \to \overline{X}'$ of E_2 results in $\overline{X}' \cong \mathbb{F}_0$. Replacing $(\overline{X}, \overline{D})$ by $(\overline{X}', \overline{D}')$, we are done. If $\widehat{X} \cong \mathbb{F}_1$, then a pivot operation at $\widehat{D}_2$ gives a log isomorphism $g : (\widehat{X}, \widehat{D}) \dashrightarrow (\widehat{X}', \widehat{D}')$ with $\widehat{X}' \cong \mathbb{F}_0$. Replacing $(\widehat{X}, \widehat{D})$ by $(\widehat{X}', \widehat{D}')$, we are done.

It remains to treat $(\widehat{X}, \widehat{D})$ of type C2. To simplify our notations, we let $(X, D) = (\widehat{X}, \widehat{D})$. We need two lemmas.

Lemma 5.4. *Let (X, D) be a log surface with X a smooth projective rational surface, $D = D_1 + D_2 + \ldots + D_n$ a circular boundary and $K_X + D = 0$. If there are $n - 1$ components $D_2, D_3, \ldots, D_n$ of D such that the intersection matrix of $\{D_2, D_3, \ldots, D_n\}$ is negative definite, then (X, D) is a genuine log K3 surface.*

Proof. If (X, D) is not a genuine log K3 surface, we have $q(X, D) \neq 0$. By (2.1), $D_1, D_2, \ldots, D_n$ are linearly dependent in $H^2(X, \mathbb{Q})$. Since $D_2, D_3, \ldots, D_n$ has negative definite intersection matrix, they are linearly independent in $H^2(X, \mathbb{Q})$. Therefore, $D_1 = a_2 D_2 + a_3 D_3 + \ldots + a_n D_n$ for some $a_i \in \mathbb{Q}$. At least one of a_i's is positive since $D_1, D_2, \ldots, D_n$ are effective. It follows that

$$\begin{aligned} 0 &> \left(\sum_{a_i > 0} a_i D_i \right)^2 + \left(\sum_{a_i > 0} a_i D_i \right) \left(\sum_{a_j < 0} a_j D_j \right) \\ &= D_1 \left(\sum_{a_i > 0} a_i D_i \right) \geq 0 \end{aligned} \tag{5.3}$$

since the intersection matrix of $\{D_2, D_3, \ldots, D_n\}$ is negative definite. Contradiction. Therefore, (X, D) is a genuine log K3 surface. □

Lemma 5.5. *Let (X, D) be a genuine log K3 surface with circular boundary $D = D_1 + D_2 + \ldots + D_n$ satisfying that $D_1^2 > 0$ and $D_i^2 \leq -2$ for $i \neq 1$. If $\operatorname{rank}_{\mathbb{Z}} \operatorname{Pic}(X) > n$, then there exists a nontrivial birational morphism $f : X \to \overline{X}$ with $\overline{D} = f_* D$ such that $(\overline{X}, \overline{D})$ is a genuine log K3 surface of type C1, C3 or with the property that*

$$\begin{aligned} &\text{there exists an irreducible component } \overline{D}_i \subset \overline{D} \\ &\text{such that } \overline{D} - \overline{D}_i \text{ has negative definite intersection matrix.} \end{aligned} \tag{5.4}$$

Proof. Note that the number $\operatorname{rank}_{\mathbb{Z}} \operatorname{Pic}(X) - \mu(D)$ remains the same after a canonical blowdown and decrease by one after a contraction of a (-1)-curve not contained in D. Suppose that D_i's satisfy (1.1).

If there is a (-1)-curve E meets D_1, we may simply blow down E and the resulting $(\overline{X}, \overline{D})$ obviously satisfies (5.4). Let us assume that

$$D_1 E = 0 \text{ for all } (-1)\text{-curves } E. \tag{5.5}$$

Clearly, $X \not\cong \mathbb{P}^2$. So there exists a fiberation $g : X \to \mathbb{P}^1$ whose general fibers are $\mathbb{P}^1$. Since $D_1^2 > 0$, $\pi_* D_1 \neq 0$. If g has a reducible fiber F_r meeting D properly, then F_r has a component E such that $D_1 E > 0$; since $E^2 < 0$ and $KE < 0$, E must be a (-1)-curve. This is impossible by (5.5). So

$$F_r \cong \mathbb{P}^1 \text{ for all fibers } F_r \text{ of } g \text{ satisfying } \dim(F_r \cap D) = 0. \tag{5.6}$$

Obviously, D_1 is either a section or a multi-section of degree 2 of g. If it is the latter, then $D_2 + D_3 + ... + D_n$ is contained in a fiber of g. Suppose that g factors through a ruled surface $\overline{X}$. Then $g_* D = \overline{D}$ is either a nodal rational curve or has two components $\overline{D} = \overline{D}_1 + \overline{D}_2$ with $\overline{D}_2^2 = 0$. Namely, $(\overline{X}, \overline{D})$ is a genuine log K3 surface of type C1 or C3.

Let us assume that D_1 is a section of g. Then there is another component D_i that is a section of g and the rest $D - D_1 - D_i$ are contained in the two fibers F_p and F_q of g over $p \neq q \in \mathbb{P}^1$. By (5.6), $F_r \cong \mathbb{P}^1$ for all $F_r \neq F_p, F_q$. Therefore, we have

$$\begin{aligned} &\operatorname{rank}_{\mathbb{Z}} \operatorname{Pic}(X) - \mu(D) = \mu(F_p) + \mu(F_q) - \mu(D) \\ &= (\mu(F_p) - \mu(F_p \cap D) - 1) + (\mu(F_q) - \mu(F_q \cap D) - 1) > 0. \end{aligned} \tag{5.7}$$

Consequently, either $\mu(F_p) \geq \mu(F_p \cap D) + 2$ or $\mu(F_q) \geq \mu(F_q \cap D) + 2$. WLOG, suppose that

$$\mu(F_p) \geq \mu(F_p \cap D) + 2. \tag{5.8}$$

It follows that F_p contains

- either one (-1)-curve E_1 and one (-2)-curve E_2 with the properties $E_2 \cap D = \emptyset$ and $E_1 E_2 = 1$
- or two disjoint (-1)-curves E_1 and E_2.

Let $\phi : X \to \overline{X}$ be the contraction of E_1 followed by a sequence of blowdowns of (-1)-curves contained in $F_p \cap D$ such that $\overline{F}_p \cap \overline{D}$ does not contain any (-1)-curves for $\overline{D} = \phi_* D$ and $\overline{F}_p = \phi_* F_p$. That is, ϕ is a birational morphism with the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\phi} & \overline{X} \\ {\scriptstyle g}\downarrow & \swarrow & \\ \mathbb{P}^1 & & \end{array} \tag{5.9}$$

such that $\overline{X}$ smooth, $\overline{F}_p \cap \overline{D}$ does not contain any (-1)-curves and the exceptional locus E_ϕ of ϕ satisfies $E_1 \subset E_\phi \subset E_1 \cup D$.

Suppose that $E_2^2 = -2$. WLOG, suppose that $E_1 D_j = 1$ for some $j > i$. Since $(\phi_* E_2)^2 \leq 0$, E_ϕ consists of at most two components. If $E_\phi = E_1$, then all components of $\overline{D}$ other than $\overline{D}_1 = \phi_* D_1$ still have self-intersections ≤ -2 and hence $(\overline{X}, \overline{D})$ satisfies (5.4). If $E_\phi = E_1 + D_j$ has two components, then $(\phi_* E_2)^2 = 0$ and we must have $\phi(E_2) = \overline{F}_p$. That is, ϕ contracts all components $F_p \cap D$. So $\overline{D}_1 \cap \overline{D}_i \cap \overline{F}_p \neq \emptyset$ for $\overline{D}_i = \phi_* D_i$. And since $E_\phi \cap D$ has one component, $\overline{D}_i^2 = D_i^2 + 1 \leq -1$. On the other hand, all components of $\overline{D}$ other than $\overline{D}_1$ and $\overline{D}_i$ still have self-intersections ≤ -2. That is, $\overline{D}_i$ is a circular boundary of type $(\lambda_1, \lambda_2, ..., \lambda_{i-1}, \overline{\lambda}_i)$ with $\lambda_k = \overline{D}_k^2 = D_k^2 \leq -2$ for $2 \leq k < i$ and $\overline{\lambda}_i = \overline{D}_i^2 \leq -1$. Therefore, the components of $\overline{D} - \overline{D}_1$ still have negative definite intersection matrix. So $(\overline{X}, \overline{D})$ satisfies (5.4) again.

Suppose that $E_2^2 = -1$. WLOG, suppose that $E_1 D_{j_1} = 1$ and $E_2 D_{j_2} = 1$ for some $j_1 \geq j_2 > i$. If $E_\phi \cap E_2 = \emptyset$, then $\overline{D}_i^2 = D_i^2 \leq -2$ and all components of $\overline{D}$ other than $\overline{D}_1$ still have self-intersections ≤ -2 and hence $(\overline{X}, \overline{D})$ satisfies (5.4). If $E_\phi \cap E_2 \neq \emptyset$, then $\phi_* E_2 = \overline{F}_p$ and ϕ contracts all components $F_p \cap D$. So $\overline{D}_1 \cap \overline{D}_i \cap \overline{F}_p \neq \emptyset$. And since $j_1 \geq j_2 > i$, $\overline{D}_i^2 = D_i^2 + 1 \leq -1$. On the other hand, all components of $\overline{D}$ other than $\overline{D}_1$ and $\overline{D}_i$ still have self-intersections ≤ -2. That is, $\overline{D}_i$ is a circular boundary of type $(\lambda_1, \lambda_2, ..., \lambda_{i-1}, \overline{\lambda}_i)$ with $\lambda_k = \overline{D}_k^2 = D_k^2 \leq -2$ for $2 \leq k < i$ and $\overline{\lambda}_i = \overline{D}_i^2 \leq -1$. Therefore, the components of $\overline{D} - \overline{D}_1$ still have negative definite intersection matrix. So $(\overline{X}, \overline{D})$ satisfies (5.4) again. □

Now we can complete the proof of Theorem 5.2.

Suppose that $D = D_1 + D_2 + ... + D_n$ is a circular boundary of type $(\lambda_1, \lambda_2, ..., \lambda_n)$ with $D_i^2 = \lambda_i \leq -2$ for all $i \neq 1$. We argue by induction on $\operatorname{rank}_{\mathbb{Z}} \operatorname{Pic}(X)$.

If $\operatorname{rank}_{\mathbb{Z}} \operatorname{Pic}(X) > n$, we may apply Lemma 5.5 to reduce $\operatorname{rank}_{\mathbb{Z}} \operatorname{Pic}(X)$ by 1. If $\operatorname{rank}_{\mathbb{Z}} \operatorname{Pic}(X) = n$, we may contract the rod $D_2 + D_3 + ... + D_n$ to obtain a log del Pezzo surface $\overline{X}$ of Picard rank 1. Indeed, we can be more precise about the singularity $\overline{p}$: $\overline{X}$ has a cyclic quotient singularity at $\overline{p}$ given by $\mathbb{C}^2/(\exp(2\pi i/a), \exp(2\pi i b/a))$, where

$$\frac{a}{b} = -\lambda_2 + \cfrac{1}{\lambda_3 + \cfrac{1}{\lambda_4 + ...}}. \tag{5.10}$$

5.3. *Proof of Theorem 5.3*

It suffices to blow up each Iitaka model $\overline{X}$ at some smooth points of $\overline{D}$, called *half point attachments* by Iitaka, such that the resulting (X, D) is a log K3, i.e., $h^0(\Omega_X(\log D)) = 0$, and (X, D) fails (4.1) (see Fig. 4). If $h^0(\Omega_{\overline{X}}(\log \overline{D})) = r$, we need to blow up r points, i.e., attach r half points.

For $\mathrm{II_{b\text{-}i}}$, it suffices to blow up $\overline{X} \cong \mathbb{P}^2$ at one point on H_1 and one point on H_2. Then (X, D) fails (4.1) since all three components of D have self-intersections ≥ 0.

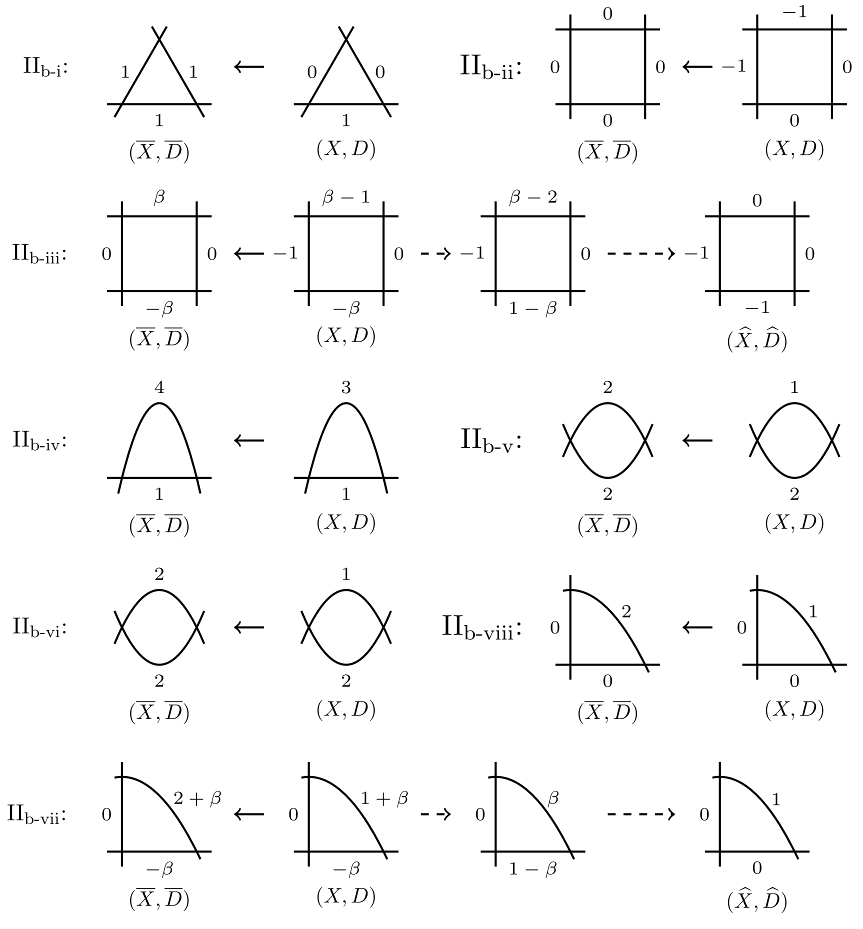

Fig. 4. Log K3 surfaces of type $\mathrm{II}_{\text{b-i}}$–$\mathrm{II}_{\text{b-viii}}$ without infinitely many $\mathbb{A}^1$ curves.

For $\mathrm{II}_{\text{b-ii}}$, it suffices to blow up $\overline{X} \cong \mathbb{F}_0$ at one point on H_1 and one point on G_1. Then (X, D) fails (4.1) since all four components of D have self-intersections ≥ -1.

For $\mathrm{II}_{\text{b-iii}}$, it suffices to blow up $\overline{X} \cong \mathbb{F}_\beta$ at one point on F_1 and one point on Δ_λ. After a sequence of pivot operations at F_2, we arrive at a log isomorphism $(X, D) \dashrightarrow (\widehat{X}, \widehat{D})$, where $\widehat{D}$ has four components with self-intersections $-1, -1, 0, 0$, respectively, and (4.1) fails.

For $\mathrm{II}_{\text{b-iv}}$, it suffices to blow up $\overline{X} \cong \mathbb{P}^2$ at one point on C. Then (X, D) fails (4.1) since both components of D have self-intersections ≥ 1 (see Fig. 2).

For $\mathrm{II}_{\text{b-v}}$, it suffices to blow up $\overline{X} \cong \mathbb{F}_0$ at one point on C_1. Then (X, D) fails (4.1) since both components of D have self-intersections ≥ 1.

For $\mathrm{II}_{\text{b-vi}}$, it suffices to blow up $\overline{X} \cong \mathbb{F}_2$ at one point on Δ_0. Then (X, D) fails (4.1) since both components of D have self-intersections ≥ 1.

For $\text{II}_{\text{b-vii}}$, it suffices to blow up $\overline{X} \cong \mathbb{F}_\beta$ at one point on C_3. After a sequence of pivot operations at F, we arrive at a log isomorphism $(X, D) \dashrightarrow (\widehat{X}, \widehat{D})$, where $\widehat{D}$ has three components with self-intersections $0, 0, 1$, respectively, and (4.1) fails.

For $\text{II}_{\text{b-viii}}$, it suffices to blow up $\overline{X} \cong \mathbb{F}_0$ at one point on C. Then (X, D) fails (4.1) since all three components of D have self-intersections ≥ 0.

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