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STRONG APPROXIMATION OVER FUNCTION FIELDS

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Abstract

By studying $\mathbb{A}^1$-curves on varieties, we propose a geometric approach to the strong approximation problem over function fields of complex curves. We prove that strong approximation holds for smooth, low degree affine complete intersections with smooth boundary at infinity.

1. Introduction

Given a variety X over a number field, the existence of rational points (integral points) and their distributions (Hasse's principle, weak approximation, and strong approximation) are extensively studied by number theorists. In general, these problems are very difficult and lacking of complete solutions.

Those questions have been studied over function fields over the past fifteen years. Let X be a smooth projective variety defined over the function field K of a complex algebraic curve. If X is rationally connected [Cam92, KMM92], then X admits rational points [GHS03, dJS03]. If X is rationally simply connected, weak approximation holds for X; see [dJS06, Has10] for the definitions and results. Furthermore, it is expected that weak approximation holds for rationally connected varieties [HT06].

While the results above focus on the projective case, number theorists study arithmetics of open varieties such as linear algebraic groups and affine hypersurfaces as well [PR94, Sko01, HT01, CTX09]. The study of integral points over geometric function fields was initiated by Hassett-Tschinkel [HT08]. They proved that for certain log Fano varieties, integral points are dense.

This paper is an attempt to build a parallel theory of integral points on open varieties over K. The natural candidates that satisfy strong approximation are *log rationally connected* varieties, that is, varieties on which a general

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pair of points can be connected by an $\mathbb{A}^1$-curve. Examples of log rationally connected varieties include:

- interior of smooth log Fano complete intersection pairs [CZ14b];
- semisimple linear algebraic groups, more generally, sober spherical homogeneous spaces of type (b) [CZ14a];
- smooth surfaces with no global higher tensor of log 1-forms [Zhu16].

In [CZ16a], we applied log rational connectedness to give another proof of Hassett-Tschinkel's theorem [HT08] and proposed the following question.

Question 1.1 ([CZ16a])**.** Does strong approximation hold for log rationally connected varieties over K?

Before this work, affine spaces were the only examples that satisfied strong approximation [Ros02, Theorem 6.13]. In this paper, we affirm strong approximation for smooth, low degree affine complete intersections.

Notation 1.2. Let $\underline{X}$ be a smooth complete intersection in $\mathbb{P}^n$ $(n \geq 2)$ of type $(d_1, \cdots, d_c)$ with $d_i \geq 2$. Let $\underline{D} \subset \underline{X}$ be a smooth hypersurface section of degree k. We call the pair $X := (\underline{X}, \underline{D})$ a *smooth complete intersection pair of type* $(d_1, \cdots, d_c; k)$. We denote by X the log scheme associated to the pair $(\underline{X}, \underline{D})$.

Theorem 1.3. *Strong approximation holds for the interior of any smooth complete intersection pair of type* $(d_1, \cdots, d_c; 1)$ *in* $\mathbb{P}^n_K$ *with*

$$\sum_{i=1}^{c} d_i^2 \leq n.$$

Corollary 1.4. *Hasse's priniciple holds for integral points of the interior of any smooth complete intersection pair of type* $(d_1, \cdots, d_c; k)$ *in* $\mathbb{P}^n_K$ *with*

$$\sum_{i=1}^{c} d_i^2 + k^2 \leq n + 1.$$

Theorem 1.3 and Corollary 1.4 give a satisfactory answer on the existence and density of integral (rational) points for low degree affine hypersurfaces defined over $\mathbb{C}[t]$. Such results are new even for affine quadric hypersurfaces with dimension at least three. In the number theoretic set up, the analogues for affine quadrics are already nontrivial theorems.

Our results on strong approximation provide an interesting geometric application.

Corollary 1.5. *Let* $(\underline{X}, \underline{D})$ *be a smooth complete intersection pair of type* $(d_1, \cdots, d_c; k)$ *in* $\mathbb{P}^n_{\mathbb{C}}$ *with* $\sum_{i=1}^{c} d_i^2 + k^2 \leq n+1$. *Then there exists an* $\mathbb{A}^1$*-curve passing through any* m*-tuple of points on* $\underline{X} \setminus \underline{D}$.

It is known that over $\mathbb{C}$, there exist rational curves on a smooth rationally connected variety through any finite number of points. But the analogues problem for log rationally connected varieties is widely open.

1.1. Idea of the proof. In this paper, we introduce the notion of $\mathbb{A}^1$-*simple-connectedness* using the stable log map compactification. This is a key to our approach for the strong approximation conjecture over K. Our proposal is parallel to the approach of de Jong-Starr [dJS06] and Hassett [Has10] for weak approximation of rationally simply connected varieties.

Theorem 1.6. *Let K be the function field of a smooth, irreducible complex algebraic curve. Let $X = (\underline{X}, \underline{D})$ be a log smooth projective variety over K. Assume the following hold:*

(1) *$\underline{X}$ satisfies weak approximation over K.*
(2) *There exists a curve class β and a geometrically irreducible component $\underline{M}$ of the moduli space $\mathcal{A}_2(X, \beta)$ of two pointed $\mathbb{A}^1$-curves defined over K such that*
   - *a general point of $\underline{M}$ parametrizes a smoothly embedded $\mathbb{A}^1$-curve.*
   - *The 2-pointed evaluation morphism*

$$ev : \underline{M} \to \underline{X} \times \underline{X}$$

   is dominant with rationally connected geometric generic fiber.

Then strong approximation holds for the interior $\underline{X} \backslash \underline{D}$ over K.

We refer to Section 1.3 for the notation and terminologies of the above theorem. The formulation of strong approximation is defined in Section 2. If we call Condition (2) above $\mathbb{A}^1$-*simple-connectedness* (with respect to the curve class β), the theorem above states that strong approximation holds for $\mathbb{A}^1$-simply connected K-varieties if weak approximation holds. Furthermore, $\mathbb{A}^1$-simple-connectedness is a geometric condition, and only depends on the interior.

Affine spaces are the first class of examples of $\mathbb{A}^1$-simply connected varieties because any pair of points can be joined by a unique affine line.

Proposition 1.7. *Affine spaces are $\mathbb{A}^1$-simply connected. Thus strong approximation holds for affine spaces over K.* □

By studying the geometry of $\mathbb{A}^1$-conics on complete intersection, we give a bound for low degree smooth complete intersection pairs to be $\mathbb{A}^1$-simply connected.

Theorem 1.8. *Let $X := (\underline{X}, \underline{D})$ be a smooth complete intersection pair of type $(d_1, \cdots, d_c; 1)$ in $\mathbb{P}^n$. Assume that $\underline{X} \setminus \underline{D}$ is not the affine space. Denote by α the line class on $\underline{X}$. Then the general fiber of the evaluation morphism defined in* (1.3.1),

$$\underline{ev} : \underline{\mathcal{A}}_2(X, 2\alpha) \to \underline{X} \times \underline{X},$$

is a smooth complete intersection in $\mathbb{P}^n$ *of type*

$$(1, 1, \cdots, d_1 - 1, d_1 - 1, d_1, \cdots, 1, 1, \cdots, d_c - 1, d_c - 1, d_c).$$

In particular, a general fiber is rationally connected if $\sum_{i=1}^{c} d_i^2 \leq n$.

Proof of Theorem 1.3. Combining the hypothesis with the works of [dJS06, Has10], we know that weak approximation holds for X. Now Theorem 1.3 follows from Theorems 1.6 and 1.8. □

Remark 1.9. The authors thank the anonymous referee, who pointed out to us that there might be an approach to the proof of Theorem 1.8 by taking the closures of the loci of $\mathbb{A}^1$-lines and $\mathbb{A}^1$-conics in the stacks of usual stable maps. Then similar to the strategy of this paper, one needs to analyze the geometry of the compactified loci of $\mathbb{A}^1$-curves. However, such compactifications seem to be less convenient to study for (1) the deformation theory for the degenerated $\mathbb{A}^1$-curves in such compactification has not been well studied to the knowledge of the authors; and (2) when the boundary D of the target has higher degree or multiple components, such compactification will be rather singular, not even normal in general. The authors haven't worked out the details in this setting as it might require some foundational work beyond the current scope of this paper. On the other hand, the logarithmic approach provides a better control of the boundary; see [CZ16b] for an example in case of boundaries with multi-components. The authors' further study of integral points and the strong approximation problem along the logarithmic approach in more general settings is currently a work in progress.

1.2. Organization of the paper. In Section 2, we state the geometric formulation of strong approximation and prove Theorem 1.6. In Sections 3 and 4, we analyze the moduli space of $\mathbb{A}^1$-lines and $\mathbb{A}^1$-conics, and conclude the proof of Theorem 1.3. We prove Corollaries 1.4 and 1.5 in Section 5.

1.3. Notation and terminology. Capital letters such as X, Y, Z, and C, etc., are reserved for log schemes with the corresponding underlying schemes denoted by $\underline{X}$, $\underline{Y}$, $\underline{Z}$, and $\underline{C}$. For any log scheme X, denote by $X^\circ \subset X$ the open locus with the trivial log structure.

An $\mathbb{A}^1$-*map* is a genus zero stable log map with precisely one marked point with a nontrivial contact order. An $\mathbb{A}^1$-*curve* is an $\mathbb{A}^1$-map with an irreducible source curve, whose image has nontrivial intersection with the open locus of the target with the trivial log structure. We call an $\mathbb{A}^1$-curve an $\mathbb{A}^1$-*line* or an $\mathbb{A}^1$-*conic* if the curve class of the $\mathbb{A}^1$-curve is the class of a line or a conic respectively.

Recall that the stack of stable log maps, viewed as a category fibered over the category of schemes, parameterizes *minimal* stable log maps. The definition of minimality can be found in [Che14, Definition 3.5.1] in the rank one case that is needed for this paper, and more generally in [AC14, Section 4] and [GS13, Section 1.5].

For any log scheme Z, any curve class β on $\underline{Z}$, and any positive integer e, denote by $\mathcal{A}_m(Z, e\beta)$ the moduli stack of $\mathbb{A}^1$-maps to Z with curve class $e\beta$, and m markings with the trivial contact order. Then $\mathcal{A}_m(Z, e\beta)$ is a log stack with the canonical log structure. Denote by $\underline{\mathcal{A}_m}(Z, e\beta)$ the underlying stack obtained by removing the log structure of $\mathcal{A}_m(Z, e\beta)$. We have the evaluation morphism induced by the m-markings with the trivial contact order

$$\underline{ev} : \underline{\mathcal{A}_m}(Z, e\beta) \to \underline{X} \times \cdots \times \underline{X} \tag{1.3.1}$$

where the right hand side is m-copies of $\underline{X}$.

Let $\mathcal{R}_m(\underline{Z}, e\beta)$ be the moduli space of m-pointed, genus zero stable maps to $\underline{Z}$ with curve class $e\beta$.

We refer to [Kat89] for the basics of logarithmic geometry, and [Kat00, Ols07] for the canonical log structures on curves. For the detailed development of stable log maps, the reader should consult [Che14, AC14, GS13].

2. Strong approximation

2.1. The arithmetic formulation. We first recall the adelic formulation of strong (weak) approximation over function fields of curves; see [Has10].

Let B a smooth irreducible projective curve over $\mathbb{C}$ with function field $F = \mathbb{C}(B)$. For each place $v \in B$, denote by K_v the completion of K at v. Let S be a nonempty finite set of places of K, $\mathfrak{o}_{K,S}$ the ring of S-integers. Denote by $\mathbb{A}_{K,S} := \prod'_{v \in B \setminus \{S\}} K_v$ the ring of adeles over all places outside S, where the product is the restricted product, i.e. all but finite number of factors are in $\mathfrak{o}_v$. The ring $\mathbb{A}_{K,S}$ has two natural topologies: the first one is the product topology, and the second one is the adelic topology, with a basis of open sets given by $\prod_{v \notin S} R_v$ where $R_v = \mathfrak{o}_v$ for all but finitely many v.

Let U be a geometrically integral algebraic variety over K. Denote by $U(K)$ the set of K-rational points, and $U(\mathbb{A}_{K,S})$ be the restricted product $\prod'_{v \notin S} U(K_v)$. Thus, the set of adelic points $U(\mathbb{A}_{K,S})$ admits the product topology and adelic topology locally inherited from that of adelic affine spaces.

Definition 2.1. We say that *strong approximation* (respectively, *weak approximation*) holds for U if for any $S \neq \emptyset$, the inclusion

$$U(K) \to U(\mathbb{A}_{K,S})$$

is dense in the adelic topology (respectively, product topology). To be more precise, this is equivalent to saying that for any finite set T of places containing

S, any integral model $\mathcal{U}$ over $\mathfrak{o}_{K,S}$ of U, and any open set $W_v \subset U(K_v)$ under the adelic topology for each place $v \in T\backslash S$, the image of $U(K)$ via the diagonal map in

$$\prod_{v\in T\backslash S} W_v \times \prod_{v\notin T} \mathcal{U}(\mathfrak{o}_v) \text{ (respectively, } \prod_{v\in T\backslash S} W_v \times \prod_{v\notin T} U(K_v)) \tag{2.1.1}$$

is not empty. We say that *Hasse's principle holds for integral points of* U if for any $S \neq \emptyset$ and any model $\mathcal{U}$ as above,

$$\prod_{v\notin S} \mathcal{U}(\mathfrak{o}_v) \neq \emptyset \text{ implies } \mathcal{U}(\mathfrak{o}_S) \neq \emptyset.$$

The above definition does not depend on the choice of model. Strong approximation implies weak approximation. The converse also holds when U is proper over K.

2.2. The geometric formulation. The geometric setting of weak approximation has been formulated and studied in [HT06]. We next translate Definition 2.1 into the geometric setting. To apply logarithmic geometry, we would like to replace the open variety U by a proper log smooth variety X with the log trivial part U.

Definition 2.2. Let X be a smooth, proper, and log smooth variety over K. Denote by $U = X^\circ$ its log trivial open subset. A *proper model* of X is a family of log schemes:

$$\pi : \mathcal{X} \to B$$

such that

(1) B is a smooth projective curve with the trivial log structure;
(2) $\underline{\pi} : \underline{\mathcal{X}} \to \underline{B}$ is proper flat over $\underline{B}$;
(3) the generic fiber of π is X.

We say such a model is *regular* if $\underline{\mathcal{X}}$ is a smooth variety. This can always be achieved via resolution of singularities.

Proposition 2.3. *Let U be the log trivial open subset of a proper, smooth, log smooth variety X defined over K. Then strong approximation holds for U away from S is equivalent to the following statement:*

Given any proper regular model of X as in Definition 2.2, *any finite set of places $T = S \cup \{b_1, \cdots, b_k\}$ such that π is smooth and log smooth over $\mathfrak{o}_{K,T}$, any smooth points x_i in $\underline{\mathcal{X}}^{sm}_{b_i}$ for $i = 1, \cdots, k$ can be realized by a section of π which is integral* (*i.e., away from the boundary*) *over $\mathfrak{o}_{K,T}$.*

Proof. With the same notation as in (2.1), since $\mathcal{U}(\mathfrak{o}_v)$ is open in $X(K_v)$ for any $v \notin T$, we may enlarge the set T such that the integral model $\mathcal{U}$ can be embedded into a regular proper model $\mathcal{X}$ of X over $\mathfrak{o}_{K,T}$ and $\mathcal{X}$ is smooth and log smooth over $\mathfrak{o}_{K,T}$. The rest is done by the iterated blow-ups of the jet datum [HT06, 2.3], [Has10, 1.5]. □

2.3. Proof of Theorem 1.6.

Step 1. To prove the theorem, it suffices to verify the statement in Proposition 2.3. Let $\underline{V} \subset \underline{X} \times \underline{X}$ denote the open subset over which $\underline{ev}$ has geometrically irreducible, and rationally connected fibers whose general points parametrize $\mathbb{A}^1$-curves.

Step 2. By assumption, we know that X is $\mathbb{A}^1$-connected. In particular, $\underline{X}$ is rationally connected. By [GHS03, KMM92], the rational points of $\underline{X}$ over K are dense. After enlarging T, there exists a rational section

$$s : \underline{B} \to \underline{\mathcal{X}}$$

such that

- s is integral over $\mathfrak{o}_{K,T}$;
- the associated rational point, still denoted by s, is general in X.

Step 3. Since weak approximation holds over K, we may choose a general section

$$t : \underline{B} \to \underline{\mathcal{X}}$$

such that

- $t(b_i) = x_i$ for $i = 1, \cdots, k$;
- the associated rational point, still denoted by t, is general in X;
- we may assume that the point (s, t) lies in the open subset $V \subset X \times X$.

Step 4. The fiber $\underline{ev}^{-1}(s, t)$ is a geometrically irreducible rationally connected variety defined over K whose general points parametrize $\mathbb{A}^1$-curves. By [GHS03, KMM92], there exists a rational point of $\underline{ev}^{-1}(s, t)$ parametrizing a smooth embedded $\mathbb{A}^1$-curve. This rational point gives a generic $\mathbb{A}^1$-ruled surface in $\underline{\mathcal{X}}$, denoted by $\underline{\mathcal{H}} \to \underline{B}$. By construction, the surface $\underline{\mathcal{H}}$ contains:

- the section s integral over $\mathfrak{o}_{K,T}$, and
- the section t. In particular, $\underline{\mathcal{H}}$ admits a local section over b_i passing through x_i for all i.

Let T' be the place of bad reductions of $\underline{\mathcal{H}}$ outside T. Since strong approximation holds for $\mathbb{A}^1_K$ away from S [Ros02, Theorem 6.13], we can find a section $\sigma : \underline{B} \to \underline{\mathcal{H}} \to \underline{\mathcal{X}}$ such that

- $\sigma(b_i) = x_i$ for all i;
- $\sigma(b) = s(b)$ for all $b \in T'$;
- σ is integral away from $T \cup T'$.

In particular, σ is integral away from T and $\sigma(b_i) = x_i$ for all i. □

3. $\mathbb{A}^1$-lines through a general point

3.1. A deformation result.

Proposition 3.1. *Let X be a projective log smooth variety. For any curve class $\beta \in H_2(\underline{X})$ and a subscheme $B \in X^\circ$ with B either a closed point or the empty set, there are finitely many sub-varieties $\{\underline{Y}_i\}$ of X° such that if $f : (\mathbb{P}^1, \infty) \to X$ is an $\mathbb{A}^1$-curve with curve class β through B, and $f(\mathbb{P}^1 \setminus \{\infty\}) \notin \underline{Y}_i$, then f is free. In particular, an $\mathbb{A}^1$-curve through B and a general point of X° with curve class β is free.*

Proof. Denote by

$$\mathcal{A}_B^\circ(X,\beta) = \begin{cases} \mathcal{A}_0^\circ(X,\beta), & \text{if } B = \emptyset, \text{ or} \\ \underline{ev}^{-1}(B), & \text{if } B \text{ is a point,} \end{cases}$$

where $\underline{ev} : \mathcal{A}_1^\circ(X,\beta) \to \underline{X}$ is the evaluation morphism induced by the marking with the trivial contact order.

Let Z_i be the irreducible component of $\mathcal{A}_B^\circ(X,\beta)$ with the universal morphism $f_i^\circ : C_i^\circ := C_i \setminus \{\infty\} \to X$. Let

$$\underline{Y}_i = \begin{cases} \overline{f_i^\circ(C_i^\circ)}, & \text{if } f_i^\circ \text{ is not dominant, and} \\ X^\circ \setminus U_i, & \text{if } f_i^\circ \text{ is dominant,} \end{cases}$$

where $\underline{U}_i \subset X^\circ$ is an open and dense subset such that f° is smooth over $\underline{U}_i$, and all closures are taken in X°.

Consider an $\mathbb{A}^1$-curve $f : (\mathbb{P}^1, \infty) \to X$ of curve class β with $f(\mathbb{P}^1 \setminus \{\infty\}) \notin \underline{Y}_i$ for any i. Let Z_j be the component containing f. By construction, the universal morphism f_j° is dominant, and f intersects $\underline{U}_j$. Same argument as in [Kol96, Chapter II 3.10] implies that f is free. □

Corollary 3.2. *Notation and assumptions as in Proposition* 3.1*, any $\mathbb{A}^1$-curve passing through B and a very general point of X° is free.*

Proof. This follows from Proposition 3.1 by taking into account all choices of curve classes. □

3.2. $\mathbb{A}^1$-lines on smooth complete intersection pairs. Consider the smooth complete intersection pair $X = (\underline{X}, \underline{D})$ as in Notation 1.2. In this subsection, we study the evaluation morphism

$$\underline{ev} : \underline{\mathcal{A}}_1(X, \alpha) \to \underline{X}.$$

Proposition 3.3.

(1) *A general fiber of $\underline{ev}$ is smooth and projective.*
(2) *Every nonempty connected component of a general fiber is of expected dimension $n - d$ where $d = d_1 + \cdots + d_c + k$.*

Proof. The first statement follows from Proposition 3.1. Since every $\mathbb{A}^1$-map with line class in a general fiber is free, the dimension is calculated by the Euler characteristic of the pullback of the log tangent bundle.

$$c_1(TX).\alpha + \dim \underline{X} + 2 - 3 - \dim \underline{X} = n + 1 - d - 1 = n - d.$$

□

Next we would like to describe the general fiber of $\underline{ev}$ explicitly in equations. Fix a general point $x \in X^\circ$. Let $\underline{L}_x$ be the fiber over x of the evaluation morphism:

$$\underline{ev} : \underline{\mathcal{A}}_1(X, \alpha) \to \underline{X}.$$

We consider the restriction of the boundary evaluation morphism on $\underline{L}_x$:

$$b' : \underline{L}_x \to \underline{D}.$$

Proposition 3.4. *If $d = d_1 + \cdots + d_c + k \leq n$, the morphism b' is a closed immersion, and the image of $\underline{L}_x$ is an irreducible, smooth complete intersection in $\mathbb{P}^n$ of type*

$$(1, \cdots, d_1, \cdots, 1, \cdots, d_c, 1, \cdots, k).$$

Let $W = \operatorname{Spec} R$ be any affine scheme. For any scheme $\underline{Z}$, we denote by $\underline{Z}_W := \underline{Z} \times W$.

The scheme L_x is determined by its W-points:

$$L_x(W) = \{\mathbb{A}^1\text{-lines in } \underline{X}_W \text{ through } x_W\}.$$

Assume for simplicity $x = [1 : 0 : \cdots : 0] \in X^\circ$. Consider a W-point $q = [x_0 : x_1 : \cdots : x_n] \in \underline{D}(W)$. A W-line ℓ joining x_W and q can be expressed as

$$[t + x_0 : x_1 : \cdots : x_n],$$

where t is the parameter of the line and $x_i \in R$ for each i.

Let F_i be the defining equation of $\underline{X}$ for $i = 1, \cdots, c$ with $\deg F_i = d_i$. Restricting them on the line equation of l, we have

$$F_i(t + x_0, x_1, \cdots, x_n) = P_{i0} \cdot t^{d_i} + P_{i1} \cdot t^{d_i - 1} + \cdots + P_{id_i}, \tag{3.2.1}$$

where $P_{ij} \in R[x_0, \cdots, x_n]$ is a homogeneous polynomial of degree j. The condition $x \in \underline{X}$ implies that $P_{i0} = 0$. The condition $\ell \subset \underline{X}_W$ is equivalent to the vanishing of $P_{i1}, \cdots, P_{id_i}$ for each i, which gives a complete intersection of type

$$(1, 2, \cdots, d_1, \cdots, 1, 2, \cdots d_c).$$

Similarly, let G be the defining equation of $\underline{D}$. Restricting them on the line equation of ℓ, we have:

$$G(t + x_0, x_1, \cdots, x_n) = Q_0 \cdot t^{d_i} + Q_1 \cdot t^{d_i - 1} + \cdots + Q_k, \tag{3.2.2}$$

where $Q_j \in R[x_0, \cdots, x_n]$ is a homogeneous polynomial of degree j. The point x_W lying outside $\underline{D}_W$ implies that $Q_0 \neq 0$. Note that Q_k is indeed G. The condition being an $\mathbb{A}^1$-line is equivalent to the vanishing of the polynomials $Q_1, \cdots, Q_k$, i.e., a complete intersection of type

$$(1, \cdots, k).$$

Now we define a complete interesection $\underline{Z}$ in $\mathbb{P}^n_W$ defined by the following equations:

$$P_{11}, \cdots, P_{1d_1}, \cdots, P_{c1}, \cdots, P_{cd_c}, Q_1, \cdots, Q_k.$$

Since $Q_k = G$, $\underline{Z}$ is automatically a closed subscheme of $\underline{D}$.

To summarize, we proved the following.

Lemma 3.5. *The image of the morphism*

$$b' : \underline{L}_x \to \underline{D}$$

lies in $\underline{Z}$.

Proof of Proposition 3.4. It suffices to prove that $\underline{L}_x$ is isomorphic to $\underline{Z}$ under b', i.e., every W-point of $\underline{Z}$ is the image of a unique W-point of $\underline{L}_x$ under b'. This follows from the fact that any W-point of $\underline{Z}$ gives a W-family of lines via the projection:

$$pr : \mathbb{P}^n_W - \{x_W\} \to \mathbb{P}^{n-1}_W,$$

where the target is the Hilbert scheme of lines through x_W. Furthermore, such a family of lines meet the boundary exactly once, hence is a family of $\mathbb{A}^1$-lines. □

Corollary 3.6. *A general fiber of* $\underline{ev}$ *is a nonempty, irreducible, and smooth complete intersection if* X *is log Fano, or equivalently,* $d \leq n$.

Proof. This follows from Propositions 3.3 and 3.4. □

4. Moduli of $\mathbb{A}^1$-conics through two general points

For the rest of this section, we work with the following assumption.

Assumption 4.1. Let $X = (\underline{X}, \underline{D})$ be a smooth complete intersection pair in $\mathbb{P}^n_{\mathbb{C}}$ of type $(d_1, \cdots, d_c; 1)$ with $d_i \geq 2$ for each i.

The goal of this section is to study general fibers of the 2-pointed evaluation morphism

$$\underline{ev} : \underline{\mathcal{A}}_2(X, 2\alpha) \to \underline{X} \times \underline{X} \qquad (4.0.1)$$

given by the two marked points with the trivial contact order. The proof of Theorem 1.8 will be concluded at the end of this section.

For later use, denote by $\underline{F}_{(p,q)}$ the fiber of (4.0.1), and $F_{(p,q)}$ the corresponding log scheme with the minimal log structure pulled back from $\mathcal{A}_2(X, 2\alpha)$. When there is no confusion of the pair of points (p, q), we will simply write $\underline{F}$ and F, and omit the subscripts.

4.1. Smoothness of the moduli.

Lemma 4.2. *For a general pair of points $(p, q) \in X^\circ \times X^\circ$, let Δ be the closed substack of F parametrizing reducible $\mathbb{A}^1$-conics. Then every geometric point f in Δ satisfies the following properties:*

(1) *The underlying curve $\underline{C}$ over $\underline{S} = \operatorname{Spec}\mathbb{C}$ consists of three irreducible genus zero components $\underline{Z}_i$ for $i = 0, 1, 2$, with precisely two nodes z_i joining $\underline{Z}_0$ and $\underline{Z}_i$ for $i = 1, 2$.*
(2) *Each component $\underline{Z}_i$ contains a marking σ_i with the trivial contact order for $i = 1, 2$.*
(3) *$\underline{Z}_0$ has three special points given by the contact marking and two nodes.*
(4) *f contracts the component $\underline{Z}_0$ to a point $x_f := f(\underline{Z}_0) \in D$.*
(5) *The restriction $\underline{f}|_{\underline{Z}_i}$ is an embedding of two free $\mathbb{A}^1$-lines for $i = 1, 2$.*
(6) *The characteristic sheaf $\overline{M}_F|_{\underline{\Delta}}$ is a locally constant sheaf with fiber $\mathbb{N}$.*
(7) *Let $C^\sharp \to S^\sharp$ be the canonical log structure on $\underline{C} \to \underline{S}$; see* [Kat00, Ols07]. *Then the canonical morphism $\overline{M}_{S^\sharp} \to \overline{M}_S$ is fiberwise given by $\mathbb{N}^2 \to \mathbb{N}, (a, b) \mapsto a + b$.*

Proof. By Assumption 4.1, the line through a general pair of points p, q in $\underline{X}$ is not contained in $\underline{X}$. Therefore, the boundary $\underline{\Delta}$ parameterizes stable log maps with property (1) – (4). Statement (5) of $\underline{f}|_{\underline{Z}_i}$ follows from Proposition 3.1 and the general choice of p, q.

Property (6) and (7) follow from the definition of minimality as in [Che14, Construction 3.3.3], [AC14, Section 4], and [GS13, Construction 1.16]. Indeed, the minimality is defined fiberwise over each geometric point. Thus to calculate the fiber of the characteristic sheaf $\overline{M}_F|_{\underline{\Delta}}$, it suffices to study the combinatorial structure of each geometric fiber f. Following the notation in [Che14, Definition 3.3.2] and [AC14, Section 4.1.1], the marked graph of f, denoted by G, has three vertices v_i corresponding to the three components $\underline{Z}_i$ for $i = 0, 1, 2$, and two edges l_j corresponding to the two nodes z_j, oriented from v_j to v_0 with contact order 1 for $j = 1, 2$. A direct calculation following [Che14, Construction 3.3.3] shows that $\overline{M}_F|_{[f]} = \mathbb{N}$. Furthermore, by the edge equation [Che14, (3.3.2)], the canonical morphism $\overline{M}_{S^\sharp} \cong \mathbb{N}^2 \to \overline{M}_S = \overline{M}_F|_{[f]} = \mathbb{N}$ is given by $(a, b) \mapsto a + b$. □

Lemma 4.3. *For a general pair of points p, q, the fiber F is a log smooth scheme with the smooth boundary divisor $\underline{\Delta}$.*

Proof. We may assume that F is nonempty. Let $U \subset \mathcal{A}_2(X, 2\alpha)$ be the open sub-stack parameterizing free $\mathbb{A}^1$-maps with the curve class 2α. Thus, the log stack U is log smooth. Recall that a log smooth scheme with a locally free log structure has a smooth underlying scheme. This follows directly from [Kat89, Theorem (3.5)]. Lemma 4.2(6) implies that U has the locally free log structure along the boundary divisor $\underline{\Delta}$, hence $\underline{U}$ is smooth along $\underline{\Delta}$. Since p, q are general, any $\mathbb{A}^1$-map in F is either an $\mathbb{A}^1$-conic or a reducible $\mathbb{A}^1$-map described in Lemma 4.2. By Proposition 3.1, any $\mathbb{A}^1$-conic through p and q is free. Combining with Lemma 4.2(4) and (5), it implies that $F \subset U$. In particular, we have $F = ev^{-1}|_U(p, q)$. Thus, the smoothness of $\underline{U}$ implies that the morphism $\underline{ev}|_U : U \to \underline{X} \times \underline{X}$ is smooth along the boundary divisor parametrizing reducible $\mathbb{A}^1$-conics. By generic smoothness, we conclude that $\underline{F}$ is smooth. In particular, the pair $F = (\underline{F}, \underline{\Delta})$ is log smooth. ☐

4.2. A lifting property in the transversal case. We pause here to study the lifting of a special type of usual stable maps to stable log maps. Here is a slightly general result that fits our need.

Let Z be a log smooth variety over $\mathbb{C}$ with a smooth boundary divisor D. This means that the log structure $\mathcal{M}_Z$ on Z is defined by

$$\mathcal{M}_Z(\underline{V}) := \{h \in \Gamma(\mathcal{O}_{\underline{V}}) \mid h|_{\underline{V} \setminus D} \in \mathcal{O}^*_{\underline{V} \setminus D}\},$$

where $\underline{V} \subset \underline{Z}$ is an arbitrary open set in Zariski topology.

Proposition 4.4. *Consider a family of genus zero usual stable maps* $\underline{f} : \underline{C} \to \underline{Z}$ *with two markings* σ_1 *and* σ_2 *over an arbitrary base scheme* $\underline{S}$ *such that:*

(1) *The family* $\underline{C} \to \underline{S}$ *is obtained by gluing two families of smooth rational curves* $\underline{C}_1 \to \underline{S}$ *and* $\underline{C}_2 \to \underline{S}$ *along the markings* $\infty_1 \subset \underline{C}_1$ *and* $\infty_2 \subset \underline{C}_2$.
(2) *Each* $\underline{C}_i \to \underline{S}$ *has two markings* σ_i *and* ∞_i *for* $i = 1, 2$.
(3) *The restriction* $\underline{f}|_{\underline{C}_i}$ *is a family of* $\mathbb{A}^1$*-curves over* $\underline{S}$ *intersecting* D *transversally along* ∞_i *for* $i = 1, 2$.

Then there exists up to a unique isomorphism, a unique family of genus zero minimal stable log maps $\tilde{f} : \tilde{C}/S \to Z$ *such that:*

(i) *The underlying scheme of* S *is* $\underline{S}$.
(ii) *The family of stable log maps has one contact marking* ∞*, and two other markings* σ_1, σ_2 *with the trivial contact order.*
(iii) *The family of usual stable maps obtained by removing log structures on* $\tilde{f}$*, forgetting the contact marking* ∞*, and then stabilizing, is* $\underline{f}$.

Remark 4.5. One could modify the above statement by assuming that $\underline{C}_1 \to \underline{S}$ and $\underline{C}_2 \to \underline{S}$ are two families of smooth irreducible curves of genus $g_1 \geq 0$ and $g_2 \geq 0$ respectively. Then the same proof as below would imply the existence of a unique family of minimal stable log maps $\tilde{f}$ of genus $g_1 + g_2$ with the properties (i), (ii), and (iii) as in Proposition 4.4. Since this general result is not needed in this paper, we leave the verification to the readers who are interested.

We divide the proof into the following two lemmas. We first prove the local existence.

Lemma 4.6. *With notation as in Proposition* 4.4*, the existence in Proposition* 4.4 *holds locally over* S*.*

Proof. Our construction here is similar to the case of [CZ16a, Proposition 2.2] but for a family of maps. We take a family of smooth rational curves $\underline{C}_0 \to \underline{S}$ with three markings ∞, ∞'_1, and ∞'_2. Such a family is necessarily trivial, and we thus have $\underline{C}_0 \cong \mathbb{P}^1 \times \underline{S}$. We have a family of nodal rational curves

$$\underline{C}' \to \underline{S}$$

obtained by gluing $\underline{C}_i$ with $\underline{C}_0$ via the identification of the markings

$$\infty_i \cong \infty'_i. \quad \text{for } i = 1, 2.$$

Now the underlying stable map $\underline{f}$ over $\underline{S}$ lifts uniquely to the underlying stable map

$$\underline{\tilde{f}} : \underline{C}' \to \underline{Z} \tag{4.2.1}$$

over $\underline{S}$ by contracting the component $\underline{C}_0$.

Consider the projectivized normal bundle $\mathbb{P} := \mathbb{P}_D(N_{D/\underline{Z}} \oplus \mathcal{O}_D)$ with two boundary divisors D_- and D_+ corresponding to the normal bundles $N^\vee_{D/\underline{Z}}$ and $N_{D/\underline{Z}}$ respectively. Here $N_{D/\underline{Z}}$ is the normal bundle of D in $\underline{Z}$. Consider the expansion $\underline{Z}[1]$ obtained by gluing $\underline{Z}$ and $\mathbb{P}$ via the identification $D \cong D_-$. We next want to lift $\underline{\tilde{f}}$ to a stable map $\underline{\tilde{f}}' : \underline{C}' \to \underline{Z}[1]$ such that

(1) $\underline{\tilde{f}}'|_{\underline{C}_i} = \underline{\tilde{f}}_{\underline{C}_i}$ for $i = 1, 2$,

(2) the composition $\underline{C}_0 \to \mathbb{P} \to D$ is compatible with $\underline{\tilde{f}}|_{\underline{C}_0}$,

(3) $\underline{\tilde{f}}'|_{\underline{C}_0} : \underline{C}_0 \to \mathbb{P}$ is a family of a relative stable map tangent to D_+ only along ∞ with multiplicity 2 and intersecting D_- transversally only along ∞_1 and ∞_2.

Replacing S by a Zariski open subset, we may assume that the pullback $\mathbb{P}_{\underline{S}} = \underline{S} \times_D \mathbb{P}$ along $\infty_i \to D$ is a trivial family of rational curves over $\underline{S}$. Note also that the required morphism $\underline{\tilde{f}}'|_{\underline{C}_0}$ factors through $\mathbb{P}_{\underline{S}}$ with the corresponding tangency along $D_{-,\underline{S}} := \underline{S} \times_D D_-$ and $D_{+,\underline{S}} := \underline{S} \times_D D_+$. Since $\underline{C}_0 \cong \mathbb{P}^1 \times \underline{S} \to \underline{S}$ is also a trivial family, to construct $\underline{\tilde{f}}'_{\underline{C}_0}$, it suffices to select a meromorphic section on $\mathbb{P}^1$ with two simple zeros along ∞'_1 and ∞'_2, and a pole of order 2 along ∞. Such a meromorphic section clearly exists. This yields the usual stable map $\underline{\tilde{f}}'$ with the desired properties.

Finally, by [Kim09] (see also the construction of [CZ16a, Proposition 2.2]), since the usual stable map $\underline{\tilde{f}}'$ intersects the boundary transversally along ∞_i, it lifts to a unique log stable map $\tilde{f}' : C' \to Z[1]$ over a log scheme S in the sense of [Kim09]. Here $Z[1]$ is the log scheme with the underlying structure $\underline{Z}[1]$, and the canonical log structure as in [Ols03], and S has underlying structure $\underline{S}$. Since there is a natural projection of log schemes $Z[1] \to Z$, composition of this projection with $\tilde{f}'$ gives us the stable log map $\tilde{f}$ as needed. $\square$

Lemma 4.7. *The uniqueness in Proposition* 4.4 *holds.*

Proof. It suffices to show the uniqueness locally. Shrinking S, we may again assume base scheme $\underline{S} = \operatorname{Spec} R$ is affine. It suffices to verify that the lift constructed in Lemma 4.6 is unique.

Assume that we have two different liftings $\tilde{f}_1 : \tilde{C}'_1/S_1 \to Z$ and $\tilde{f}_2 : \tilde{C}'_2/S_2 \to Z$. We first notice that except for the freeness in (5), all other statements in Lemma 4.2 apply to both $\tilde{f}_1$ and $\tilde{f}_2$. In particular, the two liftings $\tilde{f}_1$ and $\tilde{f}_2$ have the same underlying stable map (4.2.1) over $\underline{S}$ constructed in the proof of Lemma 4.6.

We first compare the two stable log maps over the contracted component $\underline{C}_0$. Since the underlying morphism is uniquely determined by $\underline{f}$, it remains to study the morphisms on the level of log structures. Shrinking $\underline{S}$, we may assume that $\underline{\tilde{f}}^*\mathcal{M}_Z|_{\underline{C}_0}$ is generated by a global section δ. By Lemma 4.2(6) and further shrinking $\underline{S}$, we may assume that $\mathcal{M}_{S_i}$ is generated by a global section e_i for $i = 1, 2$. By choosing the generators appropriately, we may assume that the morphism $\tilde{f}_i^\flat|_{\underline{C}_0} : \tilde{f}_i^*\mathcal{M}_Z|_{\underline{C}_0} \to \mathcal{M}_{\tilde{C}'_i}|_{\underline{C}_0}$ on the level of log structures is given by the following:

$$\tilde{f}_i^\flat(\delta) = e_i + \log\sigma, \quad \text{for } i = 1, 2, \tag{4.2.2}$$

where σ is a meromorphic function on $\underline{C}_0$ with only poles along ∞'_1 and ∞'_2, and second order zero along ∞. Since $\underline{f}|_{\underline{C}_i}$ intersects D transversally, the contact order at both nodes are equal to 1 [Che14, Definition 3.2.6], hence σ has only simple poles along ∞'_1 and ∞'_2.

We now focus on the node ∞_i of $\underline{\tilde{C}}'$ for $i = 1, 2$. Let $\mathcal{M}_{S^\sharp}$ and $\mathcal{M}_{\tilde{C}^\sharp}$ be the canonical log structure on $\underline{S}$ and $\underline{\tilde{C}}$ associated to the family $\underline{\tilde{C}} \to \underline{S}$. Shrinking S again, we assume $\mathcal{M}_{S^\sharp}$ is generated by global sections a_1 and a_2 corresponding to smoothing nodes ∞_1 and ∞_2 respectively. By Lemma 4.2(7), the log curves $\tilde{C}_1 \to S_1$ and $\tilde{C}_2 \to S_2$ are defined by the following morphisms of log structures respectively:

$$\mathcal{M}_{S^\sharp} \to \mathcal{M}_{S_1}, \quad a_i \mapsto e_1 + \log u_i, \quad \text{for } i = 1, 2, \tag{4.2.3}$$

and

$$\mathcal{M}_{S^\sharp} \to \mathcal{M}_{S_2}, \quad a_i \mapsto e_2 + \log v_i, \quad \text{for } i = 1, 2, \tag{4.2.4}$$

where $u_i, v_i \in R^*$ are some invertible elements.

Since $\underline{\tilde{f}}|_{\underline{C}_i}$ is an embedding of a family of $\mathbb{A}^1$-curves, the two sections $\tilde{f}_1^\flat(\delta)$ and $\tilde{f}_2^\flat(\delta)$ are identified with the image of $\underline{\tilde{f}}^*(\exp(\delta))$, where $\exp(\delta) \in \mathcal{O}_{\underline{\tilde{C}}'}$ is the image of δ. This in particular means that we have a canonical identification

$$\tilde{f}_1^\flat(\delta) = \tilde{f}_2^\flat(\delta) \tag{4.2.5}$$

along the node ∞_i for $i = 1, 2$. A calculation combining (4.2.5) with (4.2.2), (4.2.3) and (4.2.4) implies that

$$u_i = v_i.$$

We thus obtain an isomorphism of log structures $\mathcal{M}_{S_1} \to \mathcal{M}_{S_2}$ induced by the correspondence $e_1 \mapsto e_2$ which fits in a commutative diagram

$$\begin{array}{ccc} & \mathcal{M}_{S^\sharp} & \\ \swarrow & & \searrow \\ \mathcal{M}_{S_1} & \longrightarrow & \mathcal{M}_{S_2} \end{array}$$

with the two skew arrows given by (4.2.3) and (4.2.3). This commutative diagram induces an isomorphism of the two log curves $\tilde{C}_1' \to S_1$ and $\tilde{C}_2' \to S_2$. In view of (4.2.2), this further induces an isomorphism of the two stable log maps $\tilde{f}_1 \cong \tilde{f}_2$. Such an isomorphism is canonical from the discussion above. □

To prove Proposition 4.4, we may first construct the log lifts locally using Lemma 4.6, then glue the local construction together using Lemma 4.7. This proves the existence of lifting. The uniqueness follows from Lemma 4.7. □

4.3. Forgetful morphism to moduli of usual stable maps. Now consider the moduli space of usual stable maps with two markings $\underline{\mathcal{R}}_2(\underline{X}, 2\alpha)$. Consider the 2-evaluation morphism

$$\underline{ev} : \underline{\mathcal{R}}_2(\underline{X}, 2\alpha) \to \underline{X} \times \underline{X} \tag{4.3.1}$$

induced by the two markings. Given a pair of points $(p, q) \in \underline{X} \times \underline{X}$, denote by $\underline{F}'_{(p,q)}$ the fiber of (4.3.1) over (p, q). When there is no danger of confusion, we will write $\underline{F}'$ instead of $\underline{F}'_{(p,q)}$. Denote by $\underline{\Delta}' \subset \underline{F}'$ the locus parameterizing maps with reducible domain curves. Recall that

Lemma 4.8 ([dJS06, Lemma 5.1]). *For a general choice of* (p, q), *the scheme* $\underline{F}'$ *is smooth with a smooth divisor* $\underline{\Delta}'$.

We then consider the forgetful morphism

$$\underline{\Phi} : \underline{\mathcal{A}}_2(X, 2\alpha) \to \underline{\mathcal{R}}_2(\underline{X}, 2\alpha)$$

obtained by sending a stable log map to its underlying stable map, forgetting the marking ∞ with nontrivial contact order, then stabilizing. This induces a forgetful morphism of the fibers

$$\underline{\phi} : \underline{F} \to \underline{F}'. \tag{4.3.2}$$

Proposition 4.9. *Fixing a general choice of* (p, q), *the forgetful morphism* $\underline{\phi}$ *is an embedding of a closed sub-scheme. Furthermore, it induces an embedding of closed sub-scheme* $\underline{\Delta} \to \underline{\Delta}'$.

Proof. Note that if a usual stable map intersects the boundary of X at a single smooth point of the source curve, then it lifts to an $\mathbb{A}^1$-map in a unique way. Thus, the morphism $\underline{F} \setminus \underline{\Delta} \to \underline{F}' \setminus \underline{\Delta}'$ is an embedding. It remains to consider around the locus of stable log maps with reducible domain curves.

By Assumption 4.1 and Proposition 4.4, the forgetful morphism $\underline{\phi}$ is injective on the level of closed points. Since by our assumptions both $\underline{F}$ and $\underline{F}'$ are smooth, it remains to verify the injectivity of the tangent map.

We fix a minimal stable log map $f : C/S \to X$ over a geometric point $\underline{S} = \operatorname{Spec} \mathbb{C} \in \underline{F}$. It suffices to consider the case that $\underline{C}$ is reducible, and denote by $f' : C'/S' \to \underline{X}$ the image of f in F'.

Consider the morphism between tangent spaces $\mathrm{d}\phi_{[f]} : T_{\underline{F},[f]} \to T_{\underline{F}',[f']}$. Recall that for any smooth variety $\underline{Y}$, the tangent bundle $T_{\underline{Y}}$ can be identified with $Hom(\operatorname{Spec} \mathbb{C}[\epsilon]/(\epsilon^2), \underline{Y})$. Now the injectivity of the tangent map follows from applying the uniqueness of Proposition 4.4 to the trivial families over $\underline{S}[\epsilon] := \operatorname{Spec} \mathbb{C}[\epsilon]/(\epsilon^2)$. □

4.4. Pullback of the boundary divisor. Denote by F' the log scheme with underlying structure $\underline{F}'$, and log structure given by the canonical one associated to the underlying curves; see [Kat00, Ols07]. We note that

Lemma 4.10. *Fix a general choice of* (p, q). *The log smooth scheme* F' *has its log structure given by the boundary divisor* $\underline{\Delta}'$.

Proof. Since the locus of $\underline{F}'$ with reducible domain curves form the smooth divisor $\underline{\Delta}'$, to show that the log structure of F' is the same as the log structure given by the smooth divisor $\underline{\Delta}'$, it suffices to verify F' is log smooth. Thus, it suffices to verify $F' \to \mathfrak{M}_{0,2}$ is log smooth, where $\mathfrak{M}_{0,2}$ is the Artin stack of genus zero pre-stable curves with two markings equipped with the canonical log structure of curves. Since the morphism $F' \to \mathfrak{M}_{0,2}$ is strict, the log smoothness is equivalent to the smoothness of the underlying maps

$\underline{F}' \to \mathfrak{M}_{0,2}$. This follows from the fact that $\underline{F}'$ parameterizes free usual stable maps. □

Proposition 4.11. *Fix a general choice of* (p, q). *There is a canonical morphism of log schemes*

$$\phi : F \to F' \tag{4.4.1}$$

compatible with $\underline{\phi}$ *in* (4.3.2). *Furthermore,* $\phi^*[\underline{\Delta}'] = 2 \cdot [\underline{\Delta}]$.

Proof. Now consider a family of minimal stable log maps $f : C/S \to X$ corresponding to an S-point of F. Let $\underline{f} : \underline{C} \to \underline{X}$ be the underlying stable map over $\underline{S}$, and $\underline{f}_1 : \underline{C}_1 \to \underline{X}$ be the image of f in $\underline{F}'$. Denote by $C^\sharp \to S^\sharp$ and $C_1^\sharp \to S_1^\sharp$ the family of log curves over $\underline{C} \to \underline{S}$ and $\underline{C}_1 \to \underline{S}$ with the canonical log structure. We first notice that there is a canonical commutative diagram of log schemes

$$\begin{array}{ccc} C^\sharp & \longrightarrow & C_1^\sharp \\ \downarrow & & \downarrow \\ S^\sharp & \longrightarrow & S_1^\sharp \end{array} \tag{4.4.2}$$

To see this, we may shrink $\underline{S}$ and put two auxiliary markings on the non-contracted component of $\underline{C}$ such that $\underline{C} \to \underline{S}$ is stable, and the components contracted by $\underline{f}$ have no auxiliary markings.

Indeed, we have a commutative diagram of log stacks

$$\begin{array}{ccc} \mathcal{C}_{0,5} & \longrightarrow & \mathcal{C}_{0,4} \\ \downarrow & & \downarrow \\ \mathcal{M}_{0,5} & \longrightarrow & \mathcal{M}_{0,4} \end{array} \tag{4.4.3}$$

where $\mathcal{C}_{0,n} \to \mathcal{M}_{0,n}$ is the universal family of genus zero stable curves with the canonical log structure. Here the horizontal arrows are obtained by forgetting a marking, and we view $\mathcal{M}_{0,5}$ as the universal curve over $\mathcal{M}_{0,4}$. Thus, the diagram (4.4.2) is induced by first pulling back (4.4.3), then removing the auxiliary markings.

Denote by $F^\sharp$ the log scheme with underlying structure $\underline{F}$, and log structure given by the canonical one of the universal curves. The above argument implies that (4.4.1) is given by the composition

$$F \to F^\sharp \to F'$$

where the first arrow removes the minimal log structure and installs the canonical log structure from the curves, and the second one is given by (4.4.2). This is compatible with (4.3.2).

Finally, to compute the pullback of $\underline{\Delta}'$, we consider the morphism $F^\sharp \to F'$. Since the boundary $\underline{\Delta}$ parameterizes curves with two nodes, we have $\overline{M}_{F^\sharp}|_{\underline{\Delta}}$ is locally constant with fibers isomorphic to $\mathbb{N}^2$. In view of (4.4.3), fiberwise over $\underline{\Delta}$, the morphism $\underline{\phi}^*\overline{M}_{F'} \to \overline{M}_{F^\sharp}$ is given by

$$\mathbb{N} \to \mathbb{N}^2, 1 \mapsto (1,1).$$

Combining with Lemma 4.2, we have the morphism $\underline{\phi}^*\overline{M}_{F'} \to \overline{M}_F$ over each geometric point of $\underline{\Delta}$ is given by

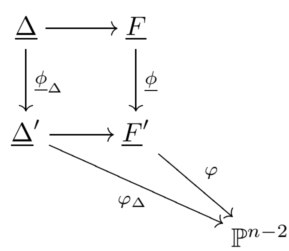
$$\mathbb{N} \to \mathbb{N}, 1 \mapsto 2.$$

Since both F and F' are log smooth with smooth boundary divisors $\underline{\Delta}$ and $\underline{\Delta}'$ respectively, this implies that $\phi^*[\underline{\Delta}'] = 2 \cdot [\underline{\Delta}]$. □

4.5. Identifying the boundary $\underline{\Delta}$ as a complete intersection. Fixing a general (p,q), the geometry of the pair $(\underline{F}', \underline{\Delta}')$ has been studied in [Pan13]. Let us recall the basic construction. Let ℓ_{pq} be the line through p, q, and let

$$\varphi : \underline{F}' \to \mathbb{P}^{n-2}$$

be the morphism sending each conic to the plane containing it. We may assume that $\mathbb{P}^{n-2}$ is the intersection of the tangent hyperplanes $T_p\underline{X}$ and $T_q\underline{X}$. Consider the following diagram:



Proposition 4.12. *The composition*

$$\underline{\Delta} \xrightarrow{u_\Delta} \underline{\Delta}' \xrightarrow{\varphi_\Delta} \mathbb{P}^{n-2} \subset \mathbb{P}^n \tag{4.5.1}$$

identifies $\underline{\Delta}$ as a complete intersection in $\mathbb{P}^n$ of type

$$(1, 1, \cdots, d_1 - 1, d_1 - 1, d_1, \cdots, 1, 1, \cdots, d_c - 1, d_c - 1, d_c, 1). \tag{4.5.2}$$

Proof. By [Pan13, Proposition 3.3], the morphism $\varphi : \underline{F}' \to \mathbb{P}^{n-2}$ is a closed embedding. It follows from Proposition 4.9 that the composition (4.5.1) is a closed embedding.

The complete intersection type follows from the $\mathbb{A}^1$-line case, and Proposition 4.4. Indeed, we define the functor

$$\mathcal{R} : Sch_{\mathbb{C}} \to Sets$$

parametrizing families of reducible conics as in the hypothesis of Proposition 4.4. By Proposition 4.4, the functor $\mathcal{R}$ is isomorphic to the functor associated to $\underline{\Delta}$ under the map φ.

Since every T-point of $\mathcal{R}$ is uniquely determined by the node of the corresponding family of conics, $\mathcal{R}$ is isomorphic to the scheme parametrizing the nodes of reducible conics. By Proposition 3.4, the locus of the boundary marking of $\mathbb{A}^1$-lines through p (or q) is a complete intersection of type

$$(1, \cdots, d_1, \cdots, 1, \cdots, d_c, 1).$$

After combining these polynomials, there is a redundancy $(d_1, \cdots, d_c, 1)$ saying that r lies on $\underline{D}$. Now the proposition follows. □

4.6. Degree of $\underline{\Delta}'$. Let $\underline{M} \subset Hilb^{2t+1}(\mathbb{P}^n)$ be the moduli space of conics in $\mathbb{P}^n$ passing through p, q. Let $\mathbb{P}^{n-2}$ be any projective subspace in $\mathbb{P}^n$ which does not intersect ℓ_{pq}. We have a canonical morphism

$$h : \underline{M} \to \mathbb{P}^{n-2}$$

which maps a conic to the plane it expands. It follows that $\underline{M}$ is a $\mathbb{P}^3$-bundle over $\mathbb{P}^{n-2}$, as the conics need to pass through two general points p, q. Let κ be the relative $\mathcal{O}(1)$-bundle of h corresponding to conics containing l_{pq}.

Lemma 4.13. *Let $\underline{M}_0$ be the open subset of $\underline{M}$ parameterizing conics which do not contain the line ℓ_{pq}. Let $\underline{\Delta}(M_0) \subset \underline{M}_0$ be the closed subset of reducible conics. Let $\underline{\Delta}(M)$ be the closure of $\underline{\Delta}(M_0)$ in $\underline{M}$. Then we have the following:*

(1) *$\underline{M}_0$ is an $\mathbb{A}^3$-bundle over $\mathbb{P}^{n-2}$.*
(2) *$\underline{\Delta}(M)$ is a smooth quadric surface bundle over $\mathbb{P}^{n-2}$.*
(3) *$\underline{\Delta}(M)$ is linearly equivalent to $2\kappa + 2h^*(\mathcal{O}_{\mathbb{P}^{n-2}}(1))$. In particular, $\underline{\Delta}(M_0)$ as a divisor in $\underline{M}_0$ is linearly equivalent to $2h^*(\mathcal{O}_{\mathbb{P}^{n-2}}(1))$.*

Proof. The first two statements follow from computation of plane conics. Indeed, fix a plane in $\mathbb{P}^n$ containing p, q. Let $p = [0 : 1 : 0]$ and $q = [0 : 0 : 1]$. A plane conic through p and q is of the form

$$a_1x^2 + a_2xy + a_3xz + a_4yz = 0.$$

It is reducible if and only if either $a_4 = 0$ or $a_1a_4 = a_2a_3$. The first case corresponds to the locus parameterizing conics containing ℓ_{pq}, while the second case does not. This proves statements (1) and (2).

From the above calculation, the divisor $\Delta(M)$ is linearly equivalent to $2\kappa + c \cdot h^*(\mathcal{O}_{\mathbb{P}^{n-2}}(1))$ for some coefficient c. To determine c, we construct a testing curve, and check its intersection number with $\Delta(M)$ as follows. We pick a general line L on $\mathbb{P}^{n-2}$ and a smooth quadric hypersurface Q in $\mathbb{P}^n$ containing p, q but not ℓ_{pq}. For any point t on L, the plane $H_{pqt} \cong \mathbb{P}^2$ spanned by the three points p, q, and t intersects Q at a conic C_t through

p, q. Furthermore, $P := \{C_t\}$ is a pencil of conics lying on the quadric surface which is the intersection of Q and the span of ℓ_{pq} and L. Therefore, there are two reducible conics in this pencil.

Since Q does not contain ℓ_{pq}, the pencil P lies in $\underline{M}_0$. In particular, $P.\kappa = 0$. On the other hand, we have $P.h^*(\mathcal{O}_{\mathbb{P}^{n-2}}(1)) = 1$ by projection formula. Hence we have $P.\Delta(M) = k$.

To finish the proof, it suffices to show that P and $\Delta(M)$ intersect transversally. Let C_{t_0} be a singular fiber of P. Since the pencil $\{C_t\}$ gives a smoothing of its singular fibers with the smooth total space, the first order deformation of P at C_{t_0} lies outside the tangent space of $\Delta(M)$. $\square$

Proposition 4.14. *The smooth divisor $\underline{\Delta}' \subset \underline{F}'$ is cut out by a homogeneous polynomial of degree two.*

Proof. This is proved in [Pan13, Proposition 5.14]. Here we present a simple proof. Consider the commutative diagram

$$\begin{array}{ccccc} \underline{\Delta}' & \longrightarrow & \underline{F}' & & \\ \downarrow & & \downarrow & \searrow^{\varphi} & \\ \underline{\Delta}(M_0) & \longrightarrow & \underline{M}_0 & \xrightarrow{h} & \mathbb{P}^{n-2} \end{array}$$

where the left square is Cartesian. Since $\underline{\Delta}(M_0) = h^*(\mathcal{O}_{\mathbb{P}^{n-2}}(2))$, so is $\underline{\Delta}'$. $\square$

4.7. Proof of Theorem 1.8.

Proposition 4.15. *The smooth divisor $\underline{\Delta} \subset \underline{F}$ is cut out by a linear form in $\mathbb{P}^{n-2}$.*

Proof. This follows from Propositions 4.11 and 4.14. $\square$

Proof of Theorem 1.8. By [Pan13, Proposition 3.3], Propositions 4.12 and 4.15, we have:

- $\underline{F} \subset \mathbb{P}^{n-2} \subset \mathbb{P}^n$ is a smooth projective variety.
- $\underline{\Delta} = W \subset \mathbb{P}^{n-2} \subset \mathbb{P}^n$ is a complete intersection of type

$$(1, 1, \cdots, d_1 - 1, d_1 - 1, d_1, \cdots, 1, 1, \cdots, d_c - 1, d_c - 1, d_c, 1).$$

- $\underline{\Delta}$ is cut out by a linear form on $\underline{F}$.

Now the theorem follows from [Pan13, Proposition 6.1]. $\square$

5. Proof of Theorem 1.3 and its corollaries

Proof of Theorem 1.3. Since affine spaces satisfy strong approximation [Ros02, Theorem 6.13], it remains to prove under Assumption 4.1. By Theorem 1.8 and the hypothesis on the degree, the general fiber of the evaluation morphism

$$\underline{ev} : \underline{\mathcal{A}}_2(X, 2\alpha) \to \underline{X} \times \underline{X}$$

is a smooth Fano complete intersection. Therefore, the general fiber is rationally connected by [Cam92, KMM92]. On the other hand, we know that weak approximation holds for X [dJS06, Has10]. Now the theorem follows from Theorem 1.6. □

Proof of Corollary 1.4. For any smooth complete intersection pair $X = (\underline{X}, \underline{D})$ in $\mathbb{P}^n$ with coordinate $[x_0 : \cdots : x_n]$, we assume that

$$\underline{X} = \{F_1 = F_2 = \cdots = F_c = 0\}$$

where $\deg F_i = d_i$ and the boundary $\underline{D} = \{G = 0\}$ where $\deg G = k$. The universal cover of $X \smallsetminus D$ can be constructed in $\mathbb{P}^{n+1}$ with the coordinate $[x_0 : \cdots : x_n : y]$ by taking the complete intersection

$$\underline{Y} = \{F_1 = \cdots = F_c = y^k - G = 0\} \tag{5.0.1}$$

with the boundary divisor

$$\underline{E} = \{y = 0\}. \tag{5.0.2}$$

We check that $(\underline{Y}, \underline{E})$ is a smooth complete intersection pair in $\mathbb{P}^{n+1}$ of type $(d_1, \cdots, d_c, k; 1)$. Furthermore the natural projection to $\mathbb{P}^n$ defined by $y = 0$ gives a cyclic branched cover of degree k over X. This yields the universal cover $Y \smallsetminus E \to X \smallsetminus D$. Now the corollary follows from strong approximation on $Y \smallsetminus E$ by Theorem 1.3. □

Proof of Corollary 1.5. Let $Y = (\underline{Y}, \underline{E})$ be the universal cover constructed in (5.0.1) and (5.0.2). For each given point $x_i \in X^\circ$, $i = 1, \cdots, m$, we may choose a lift of $y_i \in Y^\circ$. Since strong approximation holds for the constant family $\pi : Y \times \mathbb{P}^1 \to \mathbb{P}^1$ away from $S = \{\infty\}$ by Corollary 1.4, there exists an integral section curve C passing through $(y_1, t_1), \cdots, (y_m, t_m)$, where t_i's are distinct points on $\mathbb{P}^1 - \{\infty\}$. The projection $p_1(C)$ gives an $\mathbb{A}^1$-curve on Y passing through $y_1, \cdots, y_m$. Composing it with the map from Y to X gives the desired $\mathbb{A}^1$-curve. □

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