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$\mathbb{A}^1$-curves on affine complete intersections

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Abstract We generalize the results of Clemens, Ein, and Voisin regarding rational curves and zero cycles on generic projective complete intersections to the logarithmic setup.

Keywords Rational curves · Hypersurfaces · Log varieties · Rational equivalence · Chow groups

Mathematics Subject Classification Primary 14J28 · Secondary 14E05

1 Introduction

In this paper, we work with varieties over the complex numbers. First we introduce the notion of *smooth complete intersection pairs*.

Definition 1.1 Let X be a complete intersection in $\mathbb{P}^n$ of type $(d_1, \cdots, d_c)$. Let $D \subset X$ be a hypersurface section of degree k. We call the pair (X, D) a *smooth complete*

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intersection pair of type $(d_1, \cdots, d_c; k)$ if both X and D are smooth. We define the *total degree* d of the pair (X, D) by

$$d = d_1 + \cdots + d_c + k.$$

When $k = 0$, the boundary is empty and we simply denote (X, D) by X.

The existence of rational curves, algebraic hyperbolicity and rational equivalence of zero cycles on generic complete intersection of general type has been studied by Clemens [3], Ein [6,7], and Voisin [11–14].

Theorem 1.2 [Clemens, Ein, Voisin] *Let X be a generic complete intersection in $\mathbb{P}^n$ of type $(d_1, \cdots, d_c)$.*

(1) *If $d \geq 2n - c$, X has no rational curves;*
(2) *If $d \geq 2n - c + 1$, X is algebraically hyperbolic;*
(3) *If $d \geq 2n - c + 2$, no two points of X are rationally equivalent.*

The bounds above are not optimal. Voisin [13,14] further improved the bound (1) to $d \geq 2n - 2$ in case of hypersurfaces, which is optimal because hypersurfaces of degree $\leq 2n - 3$ always admit lines. Recently, Riedl and Yang further proved that X contains lines but no other rational curves if $(3n + 1)/2 \leq d \leq 2n - 3$ [10].

In this paper, we generalize Theorem 1.2 to smooth complete intersection pairs, where we study $\mathbb{A}^1$-curves and $\mathbb{A}^1$-equivalence of zero cycles instead. See Theorems 1.3, 1.6 and Corollary 1.5 below. They specialize to Theorem 1.2 when the boundary is empty.

1.1 $\mathbb{A}^1$-curves

An $\mathbb{A}^1$-*curve* is an algebraic map from $\mathbb{A}^1$ to the interior of a pair. When the boundary is empty, $\mathbb{A}^1$-curves are simply rational curves. We first study $\mathbb{A}^1$-curves on generic complete intersection pairs of general type.

Theorem 1.3 *Let (X, D) be a generic complete intersection pair in $\mathbb{P}^n$ of type $(d_1, \cdots, d_c; k)$. If $d \geq 2n - c$, the interior $X - D$ contains no $\mathbb{A}^1$-curves.*

When the boundary is nonempty, the bound in Theorem 1.3 is optimal because a general such pair in $\mathbb{P}^n$ with $d \leq 2n - c - 1$ always admits an $\mathbb{A}^1$-line. Furthermore, we complete the last step in the study of $\mathbb{A}^1$-curves on complete intersection surface pairs in $\mathbb{P}^n$ of total degree d, summarized as the table below.

$\dim X = 2$	(X, D)	$\mathbb{A}^1$-curves
$d \leq n$	Log Fano	Log rationally connected [4]
$d = n + 1$	Log K3	Generically countable [1,5,8]
$d \geq n + 2$	Of log general type	Generically none (Theorem 1.3)

1.2 Algebraic hyperbolicity

Theorem 1.4 *Let (X, D) be a generic complete intersection pair in $\mathbb{P}^n$ of type $(d_1, \cdots, d_c; k)$. If*

$$d \geq 2n - c - l + 1,$$

every closed subvariety Y of $X - D$ of dimension l has an effective log canonical bundle on its desingularisation; and if the equality is strict, Y has a big log canonical bundle on its desingularisation.

Theorem 1.4 implies algebraic hyperbolicity of such pairs.

Corollary 1.5 *Let X be a generic complete intersection in $\mathbb{P}^n$ of type $(d_1, \cdots, d_c; k)$. If $d \geq 2n - c + 1$, the interior $X - D$ is algebraically hyperbolic.*

For generic complete intersection pairs of type $(1; k)$, Theorems 1.3, 1.4 and Corollary 1.5 are proved by the first named author [2] and Pacienza-Rousseau [9].

1.3 $\mathbb{A}^1$-equivalence of zero cycles

For open varieties, the right substitution for Chow group of zero cycles is Suslin's 0-th homology group $h_0(U)$, that is, the group of zero cycles modulo $\mathbb{A}^1$-equivalences. When the boundary is empty, it coincides with the Chow group of zero cycles. For surface pairs, the log version of Mumford's theorem and Bloch's conjecture was studied in [15,16]. For generic complete intersection pairs, we have the following stronger version of Theorem 1.3.

Theorem 1.6 *Let (X, D) be a generic complete intersection pair in $\mathbb{P}^n$ of type $(d_1, \cdots, d_c; k)$. If $d \geq 2n - c + 2$, no two points of the interior $X - D$ are $\mathbb{A}^1$-equivalent.*

2 Global positivity

In this section, we generalize Voisin's global positivity result [12, Prop. 1.1] for smooth complete intersection pairs. For the rest of the paper, we fix the following notations.

Notation 2.1 With the same notations as in Definition 1.1, let $k := d_{c+1}$. Let $S^{d_i} := H^0(\mathcal{O}_{\mathbb{P}^n}(d_i))$ for $i = 1, \cdots, c+1$. Let S be the product $\prod_i^{c+1} \mathbb{P}(S^{d_i})^\vee$ of projective spaces. We denote by S° the open subset of S parametrizing smooth complete intersection tuples. Let $(\mathcal{X}, \mathcal{D}) \subset \mathbb{P}^n \times S^\circ$ be the universal family of smooth complete intersection pair. Let $\mathcal{O}_\mathcal{X}(1)$ be the pullback line bundle $pr_1^*(\mathcal{O}_{\mathbb{P}^n}(1))$. For any $t \in S^\circ$, denote by (X_t, D_t) the smooth complete intersection pair parametrized by t. We assume that $\dim X_t \geq 2$. For any log pair (Y, E), denote by $T_{Y^\dagger}$ its log tangent bundle $T_Y(-\log E)$.

Lemma 2.2 *For all* $0 < i < \dim X$ *and all smooth complete intersection pairs* (X, D) *with* $\dim X \geq 2$*, we have*

$$H^0(\Omega^i_X(\log D)) = 0.$$

Proof The long exact sequence of the residue sequence gives

$$H^0(\Omega^i_X) \to H^0(\Omega^i_X(\log D)) \to H^0(\Omega^{i-1}_D) \to H^1(\Omega^i_X).$$

The first term vanishes by the Lefschetz hyperplane theorem. If $\dim D \geq 2$, the third term vanishes by the Lefschetz hyperplane theorem as well. If $\dim D = 1$, the last map is the Gysin map which is injective. Therefore, $H^0(\Omega^i_X(\log D)) = 0$. □

Lemma 2.3 *If* $d \geq n+2$*, then* $h^0(T_{X_t^\dagger}(1)) = 0$ *for every* $t \in S^\circ$.

Proof By the isomorphism

$$T_{X_t^\dagger} \cong \Omega^{n-1}_{X_t}(\log D_t) \otimes \mathcal{O}_{X_t}(-K_{X_t} - D_t)$$

and Lemma 2.2, we have

$$\begin{aligned} h^0(T_{X_t^\dagger}(1)) &= h^0(\Omega^{n-1}_{X_t}(\log D_t) \otimes \mathcal{O}_{X_t}(-K_{X_t} - D_t) \otimes \mathcal{O}(1)) \\ &= h^0(\Omega^{n-1}_{X_t}(\log D_t) \otimes \mathcal{O}(n+2-d)) \\ &\leq h^0(\Omega^{n-1}_{X_t}(\log D)) = 0. \end{aligned}$$

□

Proposition 2.4 *The log tangent bundle*

$$T\mathcal{X}^\dagger(1)|_{X_t}$$

is globally generated for every $t \in S^\circ$ *if* $h^0(T_{X_t^\dagger}(1)) = 0$.

Proof By [4, Lem. 4.1], we have the short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathcal{D}} \longrightarrow T\mathcal{X}^\dagger|_{\mathcal{D}} \longrightarrow T_{\mathcal{D}} \longrightarrow 0.$$

The global generation of $T\mathcal{X}^\dagger(1)|_{X_t}$ implies the global generation of $T\mathcal{D}(1)|_{D_t}$. In particular, Proposition 2.4 for the nonempty boundary case implies the empty boundary case. For the rest of the proof, we assume that the boundary is nonempty.

Since $(\mathcal{X}, \mathcal{D})$ is a log smooth family over S°, we have

$$0 \longrightarrow T_{X_t^\dagger}(1) \longrightarrow T\mathcal{X}^\dagger(1)|_{X_t} \longrightarrow T_{S,t} \otimes \mathcal{O}_{X_t}(1) \longrightarrow 0.$$

By [4, Lemma 2.1], the log tangent bundle $T\mathcal{X}^\dagger$ is determined by the short exact sequence:

$$0 \longrightarrow T\mathcal{X}^\dagger \longrightarrow \mathcal{O}_{\mathcal{X}}(1)^{\oplus(n+1)} \xrightarrow{\alpha} \sum_{i=1}^{c+1} \mathcal{O}_{\mathcal{X}}(d_i) \longrightarrow 0,$$

where α is given by the multiplication of the Jacobian. The above two sequences lead to the commutative diagram:

$$\begin{array}{ccccccccc} 0 & \to & T_{X_t^\dagger}(1) & \to & T\mathcal{X}^\dagger(1)|_{X_t} & \to & T_{S,t}\otimes\mathcal{O}_{X_t}(1) & \to & 0 \\ & & \downarrow{\scriptstyle id} & & \downarrow & & \downarrow{\scriptstyle ev} & & \\ 0 & \to & T_{X_t^\dagger}(1) & \to & \mathcal{O}_{X_t}(2)^{\oplus(n+1)} & \xrightarrow{\alpha} & \sum_{i=1}^{c+1}\mathcal{O}_{\mathcal{X}}(d_i+1)|_{X_t} & \to & 0. \end{array}$$

Since $h^0(T_{X_t^\dagger}(1)) = 0$, we obtain the corresponding long exact sequences

$$\begin{array}{ccccccccc} 0 & \to & H^0(T\mathcal{X}^\dagger(1)|_{X_t}) & \to & T_{S,t}\otimes S^1 & \xrightarrow{\mu} & H^1(T_{X_t^\dagger}(1)) & \to & H^1(T\mathcal{X}^\dagger(1)|_{X_t}) \\ & & \downarrow & & \downarrow{\scriptstyle ev} \searrow{\scriptstyle \mu} & & \downarrow{\scriptstyle id} & & \downarrow \\ 0 & \to & H^0(\mathcal{O}_{X_t}(2)^{\oplus(n+1)}) & \xrightarrow{\alpha} & \prod_{i=1}^{c+1} S^{d_i+1}|_{X_t} & \to & H^1(T_{X_t^\dagger}(1)) & \xrightarrow{\beta} & H^1(\mathcal{O}_{X_t}(2)^{\oplus(n+1)}). \end{array}$$

We have the following properties:

(1) $H^0(T\mathcal{X}^\dagger(1)|_{X_t}) = \ker(\mu)$;
(2) $\ker(\beta) = \prod_i^{c+1} S^{d_i+1}/\operatorname{Im}(\alpha)$;
(3) since ev is surjective, $\operatorname{Im}(\mu) = \ker(\beta)$. Thus we have the map

$$\mu : T_{S,t}\otimes S^1 \to \ker(\beta).$$

Now for any point $x \in X_t$, tensoring all the terms in the diagrams as above with the ideal sheaf $\mathcal{I}_x$, we have another commutative diagram:

$$\begin{array}{ccccccc} H^0(T\mathcal{X}^\dagger(1)|_{X_t}\otimes\mathcal{I}_x) & \to & T_{S,t}\otimes S_x^1 & \xrightarrow{\mu_x} & H^1(T_{X_t^\dagger}(1)\otimes\mathcal{I}_x) & \to & H^1(T\mathcal{X}^\dagger(1)|_{X_t}\otimes\mathcal{I}_x) \\ \downarrow & & \downarrow{\scriptstyle ev_x} \searrow{\scriptstyle \mu_x} & & \downarrow{\scriptstyle id} & & \downarrow \\ H^0(\mathcal{O}_{X_t}(2)^{\oplus(n+1)}\otimes\mathcal{I}_x) & \xrightarrow{\alpha_x} & \prod_i^{c+1} S_x^{d_i+1} & \to & H^1(T_{X_t^\dagger}(1)\otimes\mathcal{I}_x) & \xrightarrow{\beta_x} & H^1(\mathcal{O}_{X_t}(2)^{\oplus(n+1)}\otimes\mathcal{I}_x), \end{array}$$

where $S_x^m = H^0(\mathcal{O}_{X_t}(m)\otimes\mathcal{I}_x)$. We have the following properties:

(1) $H^0(T\mathcal{X}^\dagger(1)|_{X_t}\otimes\mathcal{I}_x) = \ker(\mu_x)$;
(2) $\ker(\beta_x) = \prod_i^{c+1} S_x^{d_i+1}/\operatorname{Im}(\alpha_x)$;

(3) since ev_x is surjective (a crucial fact), we have $\operatorname{Im}(\mu_x) = \ker(\beta_x)$. Thus we write

$$\mu_x : T_{S,t} \otimes S_x^1 \to \ker(\beta_x).$$

Finally, consider the commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(T_{X_t^\dagger}(1)|_x) & \longrightarrow & H^1(T_{X_t^\dagger}(1) \otimes \mathcal{I}_x) & \longrightarrow & H^1(T_{X_t^\dagger}(1)) & \longrightarrow & 0 \\
\downarrow & & \downarrow & \overset{\gamma}{\searrow} & \downarrow{\scriptstyle \beta_x} & & \downarrow{\scriptstyle \beta} & & \\
H^0(\mathcal{O}_{X_t}(2)^{\oplus(n+1)}) & \xrightarrow{u} & H^0(\mathcal{O}_{X_t}(2)^{\oplus(n+1)}|_x) & \longrightarrow & H^1(\mathcal{O}_{X_t}(2)^{\oplus(n+1)} \otimes \mathcal{I}_x) & \longrightarrow & H^1(\mathcal{O}_{X_t}(2)^{\oplus(n+1)}) & \longrightarrow & 0.
\end{array}$$

Since $\mathcal{O}_{X_t}(2)^{\oplus n+1}$ is globally generated, the map u is surjective. In particular, the composite map γ is the zero map. Hence we get

$$0 \longrightarrow H^0(T_{X_t^\dagger}(1)|_x) \longrightarrow \ker(\beta_x) \longrightarrow \ker(\beta) \longrightarrow 0.$$

It follows that

$$\dim \operatorname{Im}(\mu_x) - \dim \operatorname{Im}(\mu) = \dim X_t.$$

On the other hand, we have

$$\begin{aligned}
\dim \ker(\mu) - \dim \ker(\mu_x) &= \dim T_{S,t} \otimes S^1 - \dim T_{S,t} \otimes S_x^1 \\
&\quad + \dim \operatorname{Im}(\mu_x) - \dim \operatorname{Im}(\mu) \\
&= \dim S + \dim X_t = \dim \mathcal{X}.
\end{aligned}$$

Thus

$$h^0(T\mathcal{X}^\dagger(1)|_{X_t}) - h^0(T\mathcal{X}^\dagger(1)|_{X_t} \otimes \mathcal{I}_x) = \dim \mathcal{X}.$$

In particular, $T\mathcal{X}^\dagger(1)|_{X_t}$ is globally generated. □

Now Proposition 2.4 implies

Corollary 2.5 *For all $l \geq 0$, the bundle $\wedge^l T\mathcal{X}^\dagger \otimes \mathcal{O}_{X_t}(l)$ is globally generated and the bundle $\wedge^l T\mathcal{X}^\dagger \otimes \mathcal{O}_{X_t}(l+1)$ is very ample if $d \geq n+2$.* □

Corollary 2.6 *If $d \geq n+2$, then $\Omega_{\mathcal{X}^\dagger}^{\dim \mathcal{X} - l}|_{X_t} = \wedge^{\dim \mathcal{X} - l}\Omega_{\mathcal{X}}(\log \mathcal{D})|_{X_t}$ is globally generated when $d \geq l + n + 1$ and is very ample when the inequality is strict.*

Proof By the isomorphism

$$\wedge^l T\mathcal{X}^\dagger \cong \wedge^{\dim \mathcal{X} - l}\Omega_{\mathcal{X}^\dagger} \otimes K_{\mathcal{X}^\dagger}^{-1}$$

we have

$$\begin{aligned}\wedge^l T\mathcal{X}^\dagger \otimes \mathcal{O}_{X_t}(l) &= \Omega_{\mathcal{X}^\dagger}^{\dim \mathcal{X}-l} \otimes K_{\mathcal{X}^\dagger}^{-1} \otimes \mathcal{O}_{X_t}(l)\\ &= \Omega_{\mathcal{X}^\dagger}^{\dim \mathcal{X}-l} \otimes \mathcal{O}_{X_t}(l+n+1-d).\end{aligned}$$

Now the assertions follow from Corollary 2.5. □

3 Proof of Main Theorems

3.1 Proof of Theorems 1.3, 1.4

Proof of Theorem 1.3 The $\mathbb{A}^1$ curves lying on the fibers of $\mathcal{X} - \mathcal{D}$ are parameterized by a subscheme of the relative Hilbert schemes of $\mathcal{X}/S^\circ$, which is a locally noetherian scheme. It has at most countably many irreducible components. If the statement of the theorem fails, one of the components of this scheme must dominate S° and the $\mathbb{A}^1$ curves parameterized by this component will cover $\mathcal{X}$; thus there exists a family of $\mathbb{A}^1$-curves

$$\begin{array}{ccc} V \times \mathbb{A}^1 & \xrightarrow{f} & (\mathcal{X}, \mathcal{D}) \\ \downarrow & & \downarrow \\ V & \xrightarrow{j} & S^\circ, \end{array}$$

where j is an étale dominant morphism and f is dominant. By [16, Lem. 3.1], the morphism f extends to a morphism of log pairs

$$f : (V' \times \mathbb{P}^1, V' \times \{\infty\}) \to (\mathcal{X}, \mathcal{D}),$$

where V' is a dense open subset of V. Here the dimension of $(V' \times \mathbb{P}^1, V' \times \{\infty\})$ is $\dim \mathcal{X} - (n-c) + 1$. Corollary 2.6 with $l = n - c - 1$ implies that $\Omega_{\mathcal{X}^\dagger}^{\dim \mathcal{X}-l}|_{X_t}$ is globally generated if

$$d \geq n - c - 1 + n + 1 = 2n - c.$$

Pullback via f gives a nontrivial section of the log canonical bundle of $(V' \times \mathbb{P}^1, V' \times \{\infty\})$, which is absurd because the pair is log uniruled. □

Proof of Theorem 1.4 The bound $d \geq 2n - c - l + 1$ implies that $\Omega_{\mathcal{X}^\dagger}^{\dim \mathcal{X}-l}|_{X_t}$ is very ample. Now Theorem 1.4 follows from the same proof as in [9, Cor. 3]. □

3.2 $\mathbb{A}^1$-equivalence of two points

To prove Theorem 1.6, we introduce a Mumford type invariant δZ following Voisin's approach [11].

Notation 3.1 Let

$$\pi : (\mathcal{X}, \mathcal{D}) \to B$$

be a log smooth family of log pairs of relative dimension $n \geq 2$ with connected fibers, B smooth of dim $B = N$ and $\mathcal{D} \neq \emptyset$. Assume that there are two distinct sections

$$p, q : B \to \mathcal{X} - \mathcal{D}$$

and denote the relative zero cycle $Z = p(B) - q(B)$.

The relative zero cycle Z defines a map

$$\delta Z : \pi_* \Omega^N_{\mathcal{X}} \xrightarrow{dp-dq} \Omega^N_B$$

where dp and dq are differential maps induced by $p : B \to \mathcal{X}$ and $q : B \to \mathcal{X}$, respectively. Since Z is disjoint from $\mathcal{D}$, δZ can be actually defined as

$$\delta Z : \pi_* \Omega^N_{\mathcal{X}}(\log \mathcal{D}) \xrightarrow{dp-dq} \Omega^N_B.$$

Therefore, δZ is a class in

$$\delta Z \in \mathrm{Hom}(\pi_* \Omega^N_{\mathcal{X}}(\log \mathcal{D}), \Omega^N_B) \subset \mathrm{Hom}(\pi_* \Omega^N_{\mathcal{X}}, \Omega^N_B).$$

If we shrink B such that $h^0(\mathcal{X}_b, \Omega^N_{\mathcal{X}}(\log \mathcal{D}))$ is constant and B is affine, then

$$\begin{aligned}
\mathrm{Hom}(\pi_* \Omega^N_{\mathcal{X}}(\log \mathcal{D}), \Omega^N_B) &= H^0((\pi_* \Omega^N_{\mathcal{X}}(\log \mathcal{D}))^\vee \otimes K_B) \\
&= H^0(R^n \pi_* (\Omega^N_{\mathcal{X}}(\log \mathcal{D})^\vee \otimes K_{\mathcal{X}/B}) \otimes K_B) \\
&= H^0(R^n \pi_* (\Omega^N_{\mathcal{X}}(\log \mathcal{D})^\vee \otimes K_{\mathcal{X}})) \\
&= H^0(R^n \pi_* (\Omega^n_{\mathcal{X}}(\log \mathcal{D}) \otimes (K_{\mathcal{X}}(\mathcal{D}))^{-1} \otimes K_{\mathcal{X}})) \\
&= H^0(R^n \pi_* (\Omega^n_{\mathcal{X}}(\log \mathcal{D})(-\mathcal{D}))) \\
&= H^n(\Omega^n_{\mathcal{X}}(\log \mathcal{D})(-\mathcal{D}))
\end{aligned}$$

where we use the relative Serre duality, the pairing

$$\Omega^N_{\mathcal{X}}(\log \mathcal{D}) \otimes \Omega^n_{\mathcal{X}}(\log \mathcal{D}) \longrightarrow K_{\mathcal{X}}(\mathcal{D})$$

and Leray spectral sequence. So we may think of δZ as a class in

$$\delta Z \in H^n(\Omega^n_{\mathcal{X}}(\log \mathcal{D})(-\mathcal{D})).$$

Clearly, the definition of δZ can be extended in an obvious way to all N-dimensional cycles of the form $Z = \sum m_i p_i(B)$, where p_i are sections of $\mathcal{X}/B$ disjoint from $\mathcal{D}$.

Next we prove that δZ vanishes for any relative zero cycle that is $\mathbb{A}^1$-equivalent to 0. This is the relative version of the log Mumford theorem [16, Thm. 3.3].

Lemma 3.2 *Let Z be a relative zero cycle $\sum m_i p_i(B)$ as above, where p_i are sections $\mathcal{X}/B$ disjoint from $\mathcal{D}$ and $m_i \in \mathbb{Z}$. If Z_b is $\mathbb{A}^1$-equivalent to 0 on $\mathcal{X}_b$ for $b \in B$ general and $\pi_*\Omega^N_{\mathcal{X}}(\log \mathcal{D})$ is locally free on B, then $\delta Z = 0$.*

Proof We give a brief sketch of the proof for the sake of completeness and refer the reader to [16] for more detail. Since $\pi_*\Omega^N_{\mathcal{X}}(\log \mathcal{D})$ is locally free, $\delta Z = 0$ if and only if $\delta Z_b = 0$ at a general point $b \in B$. So we can shrink B in any way we like. We can also apply base changes to $\mathcal{X}/B$: Let $f : \widehat{B} \to B$ be a dominant and generically finite morphism with the diagram

$$\begin{array}{ccc} (\widehat{\mathcal{X}}, \widehat{\mathcal{D}}) & \xrightarrow{f} & (\mathcal{X}, \mathcal{D}) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \widehat{B} & \xrightarrow{f} & B \end{array}$$

where $\widehat{\mathcal{X}} = \mathcal{X} \times_B \widehat{B}$ and $\widehat{\mathcal{D}} = \mathcal{D} \times_B \widehat{B}$. This induces

$$\begin{array}{ccc} f^*(\pi_*\Omega^N_{\mathcal{X}}(\log \mathcal{D})) & \xrightarrow{f^*\delta Z} & f^*\Omega^N_B \\ \downarrow & & \downarrow \\ \pi_*\Omega^N_{\widehat{\mathcal{X}}}(\log \widehat{\mathcal{D}}) & \xrightarrow{\delta \widehat{Z}} & \Omega^N_{\widehat{B}} \end{array} \tag{3.1}$$

where $\widehat{Z} = f^*Z$. Over a nonempty open set of B, the columns of (3.1) are isomorphisms. So $\delta Z = 0$ if and only if $\delta \widehat{Z} = 0$, as long as $\pi_*\Omega^N_{\mathcal{X}}(\log \mathcal{D})$ is locally free. So we may replace $(\mathcal{X}, \mathcal{D}, B, Z)$ by $(\widehat{\mathcal{X}}, \widehat{\mathcal{D}}, \widehat{B}, \widehat{Z})$ under any dominant and generically finite base change $\widehat{B} \to B$.

Let $\mathcal{X}_a = \mathcal{X} \times_B \mathcal{X} \times_B \ldots \times_B \mathcal{X}$ be the a-th self product of $\mathcal{X}$ over B, $\rho_i : \mathcal{X}_a \to \mathcal{X}$ be the i-th projection of $\mathcal{X}_a$ to $\mathcal{X}$, $\mathcal{D}_a = \sum \rho_i^*\mathcal{D}$ and

$$(\overline{\mathcal{X}}_a, \overline{\mathcal{D}}_a) = (\mathcal{X}_a, \mathcal{D}_a)/\Sigma_a$$

be the quotient of $(\mathcal{X}_a, \mathcal{D}_a)$ by the symmetric group Σ_a acting on the a factors of $\mathcal{X}_a$.

A cycle $W = p_1(B) + p_2(B) + \ldots + p_a(B)$ with p_i sections of $\mathcal{X}/B$ can be regarded as a section of $\overline{\mathcal{X}}_a/B$:

$$\begin{array}{ccc} B & \xrightarrow{(p_1, p_2, \ldots, p_a)} & \mathcal{X}_a \\ & {\scriptstyle w}\searrow & \downarrow{\scriptstyle g} \\ & & \overline{\mathcal{X}}_a \end{array}$$

Assuming that $\operatorname{supp}(W) \cap \mathcal{D} = \emptyset$, we have

$$\delta W(\omega) = \sum_{i=1}^{a} w^* g_* \rho_i^* \omega \tag{3.2}$$

for all $\omega \in H^0(\Omega_{\mathcal{X}}^N(\log \mathcal{D}))$, where $g_* : \Omega_{\mathcal{X}_a}^N(\log \mathcal{D}_a) \to \Omega_{\overline{\mathcal{X}}_a}^N(\log \overline{\mathcal{D}}_a)$ is the trace map. Note that $(\overline{\mathcal{X}}_a, \overline{\mathcal{D}}_a)$ fails to be log smooth on a closed subscheme of codimension ≥ 2 and we define its log differential sheaf to be

$$\Omega_{\overline{\mathcal{X}}_a}(\log \overline{\mathcal{D}}_a) = j_* \Omega_U(\log \overline{\mathcal{D}}_a)$$

for $j : U \hookrightarrow \overline{\mathcal{X}}_a$, where U is the open set of $\overline{\mathcal{X}}_a$ over which $\overline{\mathcal{X}}_a$ is smooth and $\overline{\mathcal{D}}_a$ has normal crossings.

Since Z_b is $\mathbb{A}^1$-equivalent to 0 on $\mathcal{X}_b$ for $b \in B$ general, again by a Hilbert scheme argument, for a sufficiently large and after a base change, there exists a morphism

$$\begin{array}{ccc} B \times \mathbb{A}^1 & \xrightarrow{z} & \overline{\mathcal{X}}_a - \overline{\mathcal{D}}_a \\ & \searrow & \downarrow \\ & & B \end{array}$$

preserving the base B such that

$$Z = z_*(B \times \{0\}) - z_*(B \times \{1\}) = \sum_{i=1}^{a} p_i(B) - \sum_{i=1}^{a} q_i(B).$$

Again by shrinking B, we may extend z to

$$\begin{array}{ccc} B \times \mathbb{P}^1 & \xrightarrow{z} & \overline{\mathcal{X}}_a \\ & \searrow & \downarrow \\ & & B \end{array}$$

Let $V = B \times \mathbb{P}^1$ and V_t be the fiber of V over $t \in \mathbb{P}^1$. Then z is a log morphism $(V, V_\infty) \to (\overline{\mathcal{X}}_a, \overline{\mathcal{D}}_a)$ and thus induces differential maps

$$z^* : \Omega_{\overline{\mathcal{X}}_a}^\bullet(\log \overline{\mathcal{D}}_a) \longrightarrow \Omega_V^\bullet(\log V_\infty).$$

For all $\gamma \in H^0(\Omega_{\overline{\mathcal{X}}_a}^N(\log \overline{\mathcal{D}}_a))$,

$$\begin{aligned} z^*\gamma \in H^0(\Omega_V^N(\log V_\infty)) &= H^0(\Omega_B^N) \otimes H^0(\mathcal{O}_{\mathbb{P}^1}) \oplus H^0(\Omega_B^{N-1}) \otimes H^0(\Omega_{\mathbb{P}^1}(1)) \\ &= H^0(\Omega_B^N) \otimes H^0(\mathcal{O}_{\mathbb{P}^1}) \end{aligned}$$

and hence

$$(j_0^* - j_1^*)z^*\gamma = 0 \tag{3.3}$$

for $j_t : B = V_t \hookrightarrow V$. Therefore, using (3.2) and (3.3), we obtain

$$\delta Z(\omega) = (j_0^* - j_1^*)z^* \sum_{i=1}^{a} g_* \rho_i^* \omega = 0$$

for all $\omega \in H^0(\Omega^N_{\mathcal{X}}(\log \mathcal{D}))$. By shrinking B, we may assume that $\pi_* \Omega^N_{\mathcal{X}}(\log \mathcal{D})$ is globally generated. So $\delta Z = 0$. □

Lemma 3.3 *If $\Omega^N_{\mathcal{X}}(\log \mathcal{D})$ is very ample when restricted to a general fiber of $\mathcal{X}/B$, then $\delta Z \neq 0$ for $Z = p(B) - q(B)$ and all pairs of distinct sections p, q of $\mathcal{X}/B$ disjoint from $\mathcal{D}$.*

Proof By shrinking B, we may assume that $h^0(X_b, \Omega^N_{\mathcal{X}}(\log \mathcal{D}))$ is constant, B is affine and $\Omega^N_{\mathcal{X}}(\log \mathcal{D})$ is very ample on $\mathcal{X}$. Hence the two sections $P = p(B)$ and $Q = q(B)$ impose independent conditions on $H^0(\Omega^N_{\mathcal{X}}(\log \mathcal{D}))$. That is, we have a surjection

$$H^0(\Omega^N_{\mathcal{X}}(\log \mathcal{D})) \twoheadrightarrow H^0(P, \Omega^N_{\mathcal{X}}(\log \mathcal{D})|_P) \oplus H^0(Q, \Omega^N_{\mathcal{X}}(\log \mathcal{D})|_Q).$$

Then we see that δZ is surjective through the diagram

$$\begin{array}{ccc}
H^0(\Omega^N_{\mathcal{X}}(\log \mathcal{D})) & \twoheadrightarrow & H^0(P, \Omega^N_{\mathcal{X}}(\log \mathcal{D})|_P) \oplus H^0(Q, \Omega^N_{\mathcal{X}}(\log \mathcal{D})|_Q) \\
 & \searrow^{\delta Z} & \downarrow {\scriptstyle dp \oplus dq} \\
 & & H^0(\Omega^N_B) \oplus H^0(\Omega^N_B) \\
 & & \downarrow {\scriptstyle \omega_1 - \omega_2} \\
 & & H^0(\Omega^N_B)
\end{array}$$

So $\delta Z \neq 0$. □

Proof of Theorem 1.6 If the conclusion is not true, again by a Hilbert scheme argument as before, there exists a smooth variety S' étale dominant over an open subset of S such that

- the base change $(\mathcal{X}, \mathcal{D}) \times_S S'$ admits two distinct sections p and q over S';
- the relative zero cycle $Z = p(S') - q(S')$ is trivial under $\mathbb{A}^1$-equivalence.

By Lemma 3.2, δZ vanishes on some open subset of S'. On the other hand, by Corollary 2.6, $\Omega^{\dim \mathcal{X} - \dim X_t}_{\mathcal{X}^\dagger}|_{X_t}$ is very ample if

$$d \geq n - c + n + 1 + 1 = 2n - c + 2.$$

Thus by Lemma 3.3, δZ is not zero at a general point $t \in S'$. We have a contradiction. □

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