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Fano hypersurfaces in positive characteristic

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Abstract. We prove that a general Fano hypersurface in a projective space over an algebraically closed field is separably rationally connected.

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1. Introduction

In this paper, we work with varieties over an algebraically closed field k of arbitrary characteristic.

Definition 1 ([5, IV.3]). *Let X be a smooth variety defined over k.*

A variety X is rationally connected *if there is a family of irreducible proper rational curves $g: U \to Y$ and a morphism $u: U \to X$ such that the morphism $u^{(2)}: U \times_Y U \to X \times X$ is dominant.*

A variety X is separably rationally connected *if there exists a proper rational curve $f: \mathbb{P}^1 \to X$ such that the image lies in the smooth locus of X and the pullback of the tangent sheaf f^*TX is ample. Such rational curves are called* very free *curves.*

We refer to Kollár's book [5] or the work of Kollár–Miyaoka–Mori [6] for the background. If X is separably rationally connected, then X is rationally connected. The converse is true when the ground field is of characteristic zero by generic smoothness. In positive characteristic, the converse statement is open.

In characteristic zero, a very important class of rationally connected varieties are Fano varieties, i.e., smooth varieties with ample anticanonical bundles. In positive characteristic, we only know that they are rationally chain connected.

Question 2 (Kollár). *In arbitrary characteristic, is every smooth Fano variety separably rationally connected?*

The question is open even for Fano hypersurfaces in projective spaces. In this paper, we prove the following theorem.

Theorem 3. *In arbitrary characteristic, a general Fano hypersurface of degree n in $\mathbb{P}^n_k$ contains a minimal very free rational curve of degree n, i.e., the pullback of the tangent bundle has the splitting type $\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus(n-2)}$.*

Theorem 4. *In arbitrary characteristic, a general Fano hypersurface in $\mathbb{P}^n_k$ is separably rationally connected.*

de Jong and Starr [4] proved that every family of separably rationally connected varieties over a curve admits a rational section. Thus using Theorem 4, we give another proof of Tsen's theorem.

Corollary 5. *Every family of Fano hypersurfaces in $\mathbb{P}^n$ over a curve admits a rational section.*

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2. Typical Curves and Deformation Theory

Notation 6. Let n be an integer ≥ 3. Let X be a hypersurface of degree n in $\mathbb{P}^n$. Let C be a smooth rational curve of degree e contained in the smooth locus of X. Consider the normal bundle exact sequence.

$$0 \longrightarrow TC \longrightarrow TX|_C \longrightarrow \mathcal{N}_{C|X} \longrightarrow 0$$

By adjunction, the degree of $TX|_C$ is the degree of $\mathcal{O}_{\mathbb{P}^n}(1)|_C$. Thus the degree of the normal bundle $\mathcal{N}_{C|X}$ is $e-2$ and the rank is $n-2$.

Definition 7. *Let e be a positive integer $\leq n$. A smooth rational curve C of degree e contained in the smooth locus of X is* typical, *if its normal bundle is the following:*

$$\mathcal{N}_{C|X} \cong \begin{cases} \mathcal{O}_C^{\oplus(n-3)} \oplus \mathcal{O}_C(-1), & \text{if } e = 1, \\ \mathcal{O}_C^{\oplus(n-e)} \oplus \mathcal{O}_C(1)^{\oplus(e-2)}, & \text{if } e \geq 2. \end{cases}$$

The curve C is a typical line, *resp.,* typical conic *if moreover the degree of C is one, resp., two.*

Remark 8.

(1) For a typical line L on X, there is a canonically defined *trivial subbundle* $\mathcal{O}_L^{\oplus(n-2)}$ in $\mathcal{N}_{L|X}$.
(2) When $e = n$, typical rational curves of degree n are very free.

Lemma 9. *Let C be a smooth rational curve of degree e on the smooth locus of a hypersurface X of degree n, where $2 \leq e \leq n$. Then C is typical if and only if both of the following conditions hold:*

(1) $h^1(C, \mathcal{N}_{C|X}(-1)) = 0$,
(2) $h^1(C, \mathcal{N}_{C|X}(-2)) \leq n - e$.

Proof. Recall that the rank of the normal bundle $\mathcal{N}_{C|X}$ is $n-2$ and the degree is $e-2$. Assume that $\mathcal{N}_{C|X}$ has the splitting type $\mathcal{O}_C(a_1) \oplus \cdots \oplus \mathcal{O}_C(a_{n-2})$, where $a_1 \geq \cdots \geq a_{n-2}$. Condition (1) is equivalent to that $a_{n-2} \geq 0$. Condition (2) implies that at most $n-e$ of the a_i's are 0. By degree count, C is a typical rational curve of degree e. □

Similarly, we have the following cohomological criterion for typical lines.

Lemma 10. *Let L be a smooth line on the smooth locus of X. Then L is typical if and only if both of the following conditions hold:*

(1) $h^1(C, \mathcal{N}_{L|X}) = 0$,
(2) $h^1(C, \mathcal{N}_{L|X}(-1)) \leq 1$.

Let H_n be the Hilbert scheme of hypersurfaces of degree n in $\mathbb{P}^n$. It is isomorphic to a projective space. Let $\mathcal{X} \to H_n$ be the universal hypersurface. The morphism $\mathcal{X} \to H_n$ is flat projective and there exists a relative very ample invertible sheaf $\mathcal{O}_{\mathcal{X}}(1)$ on $\mathcal{X}$.

Let $R_{e,n}$ be the Hilbert scheme parameterizing flat projective families of one-dimensional subschemes in $\mathcal{X}$ with the Hilbert polynomial $P(d) = ed+1$. By [5, Theorem 1.4], $R_{e,n}$ is projective over H_n.

Let $\mathscr{C}$ be the universal family over $R_{e,n}$, denoted by $\pi : \mathscr{C} \to R_{e,n}$. We have the following diagram,

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{\ i\ } & \mathscr{X} \times_{H_n} R_{e,n} \\ {\scriptstyle\pi}\downarrow & \swarrow & \\ R_{e,n} & & \end{array}$$

where i is a closed immersion.

Typical rational curves on hypersurfaces are deformation open in the following sense.

Proposition 11. *Let e be a positive integer $\leq n$. There exists an open subset in $R_{e,n}$ parameterizing typical curves of degree e in hypersurfaces of degree n.*

Proof. By Lemma 10 and Lemma 9, every typical curve C on a hypersurface X gives an unobstructed point in $R_{e,n}$. By the upper semicontinuity theorem [3, III.12.8], a small deformation of (X_t, C_t) is still typical and C_t is contained in the smooth locus of X_t. □

Proposition 12. *Let X be a hypersurface of degree n in $\mathbb{P}^n$. Let L and M be two typical lines on X intersecting transversally at one point p. Assume that the following conditions hold:*

(1) *the direction T_pL lies outside the trivial subbundle of $\mathscr{N}_{M|X}$;*
(2) *the direction T_pM lies outside the trivial subbundle of $\mathscr{N}_{L|X}$.*

Then the pair $(X, D = L \cup M) \in R_{2,n}$ can be smoothed to a pair (X', C) where C is a typical conic in X'. Furthermore, there exists an open neighborhood of $(X, D = L \cup M)$ in which any smoothing of $(X, D = L \cup M)$ is a typical conic.

Proof. Let D be the union of the lines L and M. Since D is a local complete intersection in the smooth locus of X, the normal bundle $\mathscr{N}_{D|X}$ is locally free. We have the following short exact sequence.

$$0 \longrightarrow \mathscr{N}_{L|X} \longrightarrow \mathscr{N}_{D|X}|_L \longrightarrow T_pM \longrightarrow 0$$

By [2, Lemma 2.6], the locally free sheaf $\mathscr{N}_{D|X}|_L$ is the sheaf of rational sections of $\mathscr{N}_{L|X}$ which has at most one pole at the direction of T_pM. Since $\mathscr{N}_{L|X} \cong \mathscr{O}_L^{\oplus(n-3)} \oplus \mathscr{O}_L(-1)$, condition (2) implies that $\mathscr{N}_{D|X}|_L$ is isomorphic to $\mathscr{O}_L^{\oplus(n-2)}$.

By the same argument, condition (1) implies that the sheaf $\mathscr{N}_{D|X}|_M$ is isomorphic to $\mathscr{O}_M^{\oplus(n-2)}$. Now we have the following short exact sequence.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{N}_{D|X}|_M(-p) & \longrightarrow & \mathscr{N}_{D|X} & \longrightarrow & \mathscr{N}_{D|X}|_L & \longrightarrow & 0 \\ & & \| & & & & \| & & \\ & & \mathscr{O}_M(-1)^{\oplus(n-2)} & & & & \mathscr{O}_L^{\oplus(n-2)} & & \end{array}$$

First we claim that there is a smoothing of D. Since $h^1(D, \mathscr{N}_{D|X}) = 0$, the pair (X, D) is unobstructed in $R_{2,n}$, cf. [5, I.2]. By [7, Lemma 3.17], it suffices to show that the map

$$H^0(D, \mathscr{N}_{D|X}) \to H^0(L, \mathscr{N}_{D|X}|L) \to T_pM$$

is surjective. Since $H^1(M, \mathscr{N}_{D|X}|_M(-p)) = 0$, the first map is surjective. Since $H^1(L, \mathscr{N}_{D|X}|_L) = 0$, the second map is also surjective.

Let q, r be two distinct points on $L - \{p\}$. We have

$$h^1(D, \mathscr{N}_{D|X}(-q)) = 0 \quad \text{and} \quad h^1(D, \mathscr{N}_{D|X}(-q-r)) = n-2.$$

Now for any smoothing (X_t, D_t) of (X, D) over a curve T, we can specify two distinct points q_t and r_t on D_t which specialize to q and r on D. After shrinking T, we may assume that the conic

D_t is contained in the smooth locus of X_t. By the upper semicontinuity theorem and Lemma 9, D_t is a typical conic on X_t. □

Definition 13. *Let X be a hypersurface of degree n in $\mathbb{P}^n$. A* typical comb *with m teeth on X is a reduced curve in X with $m+1$ irreducible components $C, L_1, \ldots, L_m$ satisfying the following conditions:*

(1) *C is a typical conic on X;*
(2) *$L_1, \ldots, L_m$ are disjoint typical lines on X and each L_i intersects C transversally at p_i.*

The conic C is called the handle *of the comb and L_i's are called the* teeth.

Proposition 14. *Let X be a hypersurface of degree n in $\mathbb{P}^n$. Let $D = C \cup L_1 \cup \cdots \cup L_{n-2}$ be a typical comb with $n-2$ teeth on X. Let p_i be the intersection point $L_i \cap C$. Assume that the following conditions hold:*

(1) *the direction $T_{p_i}C$ lies outside the trivial subbundle of $\mathcal{N}_{L_i|X}$;*
(2) *the directions $T_{p_i}L_i$ are general in $\mathcal{N}_{C|X}$ so that the sheaf $\mathcal{N}_{D|X}|_C$ is isomorphic to $\mathcal{O}_C(1)^{\oplus(n-2)}$.*

Then the pair $(X, D) \in R_{n,n}$ can be smoothed to a pair (X', C') where C' is a very free rational curve on X'.

Proof. The proof is very similar to the proof of Proposition 12. Here we only sketch the proof. Condition (1) implies that the sheaf $\mathcal{N}_{D|X}|_{L_i}$ is isomorphic to $\mathcal{O}_{L_i}^{\oplus(n-2)}$ for each i. We have the following short exact sequence.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bigoplus_i \mathcal{N}_{D|X}|_{L_i}(-p) & \longrightarrow & \mathcal{N}_{D|X} & \longrightarrow & \mathcal{N}_{D|X}|_C & \longrightarrow & 0 \\ & & \| & & & & \| & & \\ & & \mathcal{O}_{L_i}(-1)^{\oplus(n-2)} & & & & \mathcal{O}_C(1)^{\oplus(n-2)} & & \end{array}$$

Since $H^1(D, \mathcal{N}_{D|X}) = 0$, D is unobstructed. By diagram chasing, the map $H^0(D, \mathcal{N}_{D|X}) \to \bigoplus_i T_{p_i}L_i$ is surjective. Thus we can smooth the typical comb D.

Now we may choose a smoothing (X_t, D_t) and specify two distinct points (q_t, r_t) which specialize to two distinct points (q, r) on $C - \{p_1, \ldots, p_{n-2}\}$. By the long exact sequence in cohomology associated to the above exact sequence, we know that $h^1(D, \mathcal{N}_{D|X}(-q-r)) = 0$. By the upper semicontinuity theorem, a general smoothing of the pair (X, D) gives a very free curve in a general hypersurface. □

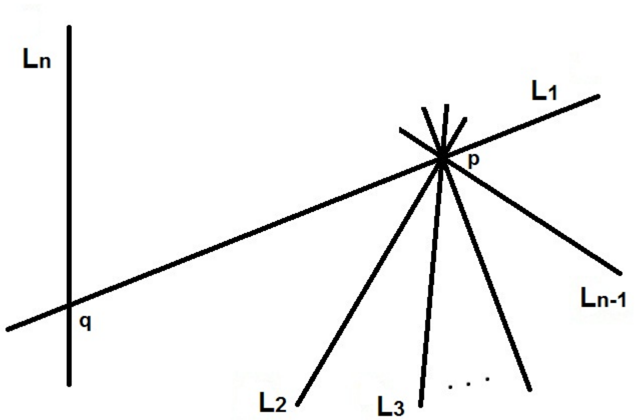
3. An Example

In this section, we construct a hypersurface of degree n in $\mathbb{P}^n$, which contains a special configuration of typical lines. Later we will use this example to produce a very free curve in a general hypersurface.

Notation 15. Let n be an integer ≥ 4. Let $[x_0 : \cdots : x_n]$ be homogeneous coordinates for $\mathbb{P}^n$. Let X be a hypersurface of degree n in the projective space $\mathbb{P}^n$ defined by the following equation $F(x_0,\ldots,x_n)$. Let e_i be the point in $\mathbb{P}^n$ represented by the i-th unit vector in k^{n+1}.

$$
\begin{array}{llll}
x_0^{n-1}x_n & +x_1^{n-3}x_n^2x_0 & +(x_1^{n-1}+x_0x_1^{n-2}+\cdots+x_0^{n-3}x_1^2)x_2 & +(x_2^{n-1}+x_0x_2^{n-2}+\cdots+x_0^{n-3}x_2^2)x_3 \\
 & +x_1^{n-4}x_n^3x_3 & +(x_0x_1^{n-2}+\cdots+x_0^{n-3}x_1^2)x_3 & +(x_0x_2^{n-2}+\cdots+x_0^{n-3}x_2^2)x_4 \\
 & \vdots & \vdots & \vdots \\
 & +x_1x_n^{n-2}x_{n-2} & +(x_0^{n-4}x_1^3+x_0^{n-3}x_1^2)x_{n-2} & +(x_0^{n-4}x_2^3+x_0^{n-3}x_2^2)x_{n-1} \\
 & +x_n^{n-1}x_{n-1} & +x_0^{n-3}x_1^2x_{n-1} & +x_0^{n-3}x_2^2x_1 \\
 & & & \\
 & +\cdots & +(x_{n-1}^{n-1}+x_0x_{n-1}^{n-2}+\cdots+x_0^{n-3}x_{n-1}^2)x_1 & \\
 & +\cdots & +(x_0x_{n-1}^{n-2}+\cdots+x_0^{n-3}x_{n-1}^2)x_2 & \\
 & \vdots & \vdots & \\
 & +\cdots & +(x_0^{n-4}x_{n-1}^3+x_0^{n-3}x_{n-1}^2)x_{n-3} & \\
 & +\cdots & +x_0^{n-3}x_{n-1}^2x_{n-2} &
\end{array}
$$

Let p be the point $[1:0:\cdots:0]$ and q be the point $[0:1:0:\cdots:0]$. Let L_i be the line spanned by $\{e_0,e_i\}$ for $i=1,\ldots,n-1$ and L_n be the line spanned by $\{e_1,e_n\}$. It is easy to check that they all lie in the hypersurface X. Let C be the union of $L_1,\ldots,L_n$. The following picture shows the configuration of the points and the lines in the projective space.



Lemma 16.

(1) *Both p and q lie in the smooth locus of X.*
(2) *The tangent space T_pX is the hyperplane $\{x_n = 0\}$, which is spanned by the lines $L_1,\ldots,L_{n-1}$.*
(3) *The tangent space of T_qX is the hyperplane $\{x_2 = 0\}$.*

Proof. By taking the partial derivatives of F, we have $\frac{\partial F}{\partial x_i}(p) = 0$ for $i = 0,\ldots,n-1$ and $\frac{\partial F}{\partial x_n}(p) = 1$. Similarly, we have $\frac{\partial F}{\partial x_i}(q) = 0$ for $i \neq 2$ and $\frac{\partial F}{\partial x_2}(q) = 1$. □

Lemma 17. *The lines $L_1,\ldots,L_{n-1}$ are in the smooth locus of X.*

Proof. We will prove the case for line L_1. The remaining cases can be computed directly by the same method. Denote $L_1 = \{[x_0 : x_1 : 0 : \cdots : 0] \in \mathbb{P}^n\}$. By restricting the partial derivatives of the defining equation of the hypersurface X on L_1, we get the following.

$$\begin{aligned} \left.\frac{\partial F}{\partial x_2}\right|_{L_1} &= x_1^{n-1} + x_0 x_1^{n-2} + \cdots + x_0^{n-3} x_1^2 \\ \left.\frac{\partial F}{\partial x_3}\right|_{L_1} &= x_0 x_1^{n-2} + \cdots + x_0^{n-3} x_1^2 \\ &\vdots \\ \left.\frac{\partial F}{\partial x_{n-2}}\right|_{L_1} &= x_0^{n-4} x_1^3 + x_0^{n-3} x_1^2 \\ \left.\frac{\partial F}{\partial x_{n-1}}\right|_{L_1} &= x_0^{n-3} x_1^2 \\ \left.\frac{\partial F}{\partial x_n}\right|_{L_1} &= x_0^{n-1} \end{aligned} \tag{1}$$

For points on L_1 with $x_0 \neq 0$, we have $\frac{\partial F}{\partial x_n}|_{L_1} \neq 0$. At the point q, $\frac{\partial F}{\partial x_2}|_{L_1} \neq 0$. Hence every point on the line L_1 is a smooth point of X. □

Lemma 18. *The line L_n is in the smooth locus of X.*

Proof. By restricting the partial derivatives of the defining equation of X on L_n, we get the following.

$$\begin{aligned} \left.\frac{\partial F}{\partial x_0}\right|_{L_n} &= x_1^{n-3} x_n^2 \\ \left.\frac{\partial F}{\partial x_3}\right|_{L_n} &= x_1^{n-4} x_n^3 \\ &\vdots \\ \left.\frac{\partial F}{\partial x_{n-2}}\right|_{L_n} &= x_1 x_n^{n-2} \\ \left.\frac{\partial F}{\partial x_{n-1}}\right|_{L_n} &= x_n^{n-1} \\ \left.\frac{\partial F}{\partial x_2}\right|_{L_n} &= x_1^{n-1} \end{aligned} \tag{2}$$

For points on L_n with $x_1 \neq 0$, we have $\frac{\partial F}{\partial x_2}|_{L_n} \neq 0$. For points on L_n with $x_n \neq 0$, we have $\frac{\partial F}{\partial x_{n-1}}|_{L_n} \neq 0$. Hence every point on the line L_n is a smooth point of X. □

Proposition 19. *With the setup as in Notation 15, X satisfies the following properties.*

(1) *The lines $L_1, \ldots, L_n$ are typical in X.*
(2) *For $i = 1, \ldots, n-1$, the trivial subbundle of the normal bundle $\mathcal{N}_{L_i|X}$ at p is generated by $\partial_{\overline{i+1}} - \partial_{\overline{i+2}}, \ldots, \partial_{\overline{i+n-3}} - \partial_{\overline{i+n-2}}$, where the notation $\bar{j}$ is j if j is less than n and $j-(n-1)$ if otherwise.*
(3) *The trivial subbundle of the normal bundle $\mathcal{N}_{L_1|X}$ at q is generated by $\partial_3, \ldots, \partial_{n-1}$*
(4) *The trivial subbundle of the normal bundle $\mathcal{N}_{L_n|X}$ at q is generated by $\partial_3, \ldots, \partial_{n-1}$.*

Proof. Let L be a line in X. We have the following short exact sequences.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{N}_{L|X}(-1) & \longrightarrow & \mathcal{N}_{L|\mathbb{P}^n}(-1) & \longrightarrow & \mathcal{N}_{X|\mathbb{P}^n}|_L(-1) & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ 0 & \longrightarrow & \mathcal{N}_{L|X}(-1) & \longrightarrow & \mathcal{O}_L^{\oplus(n-1)} & \longrightarrow & \mathcal{O}_L(n-1) & \longrightarrow & 0. \end{array}$$

The associated long exact sequence is the following.

$$H^0(L,\mathcal{N}_{L|X}(-1)) \to k^n \xrightarrow{\alpha} H^0(L,\mathcal{O}_L(n-1)) \longrightarrow H^1(L,\mathcal{N}_{L|X}(-1)) \longrightarrow 0$$

where the map α sends the natural basis of k^n to the derivatives of F with respect to the normal directions of L in $\mathbb{P}^n$. By Lemma 10, L is typical if and only if the image of α is of codimension one in $H^0(L,\mathcal{O}_L(n-1))$.

When $L = L_1$, by (1), $\frac{\partial F}{\partial x_2}|_{L_1},\dots,\frac{\partial F}{\partial x_n}|_{L_1}$ form a codimensional-one subspace of $H^0(L_1,\mathcal{O}_{L_1}(n-1))$. Thus we get that $H^1(L_1,\mathcal{N}_{L_1|X}(-1))$ is one dimensional, i.e., L_1 is typical in X.

By the short exact sequence above, $N_{L_1|X}(-1)$ is a subbundle of the trivial bundle $\mathcal{O}_{L_1}^{\oplus(n-1)}$ which maps to 0 in $\mathcal{O}_{L_1}(n-1)$. Let $\partial_2,\dots,\partial_n$ be the generators of $\mathcal{O}_{L_1}^{\oplus(n-1)}$. We get $N_{L_1|X}(-1)$ is generated by $x_0(\partial_2-\partial_3)-x_1(\partial_3-\partial_4),\dots,x_0(\partial_{n-2}-\partial_{n-1})-x_1\partial_{n-1}, x_0^2\partial_{n-1}-x_1^2\partial_n$ as an $\mathcal{O}_{L_1}$-module. If we restrict the bundle at p and q, we get properties (2) and (3) for L_1.

When $L = L_2,\dots,L_{n-1}$, we can prove properties (2) and (3) in a similar way. When $L = L_n$, (4) follows from the same computation as above by applying (2). □

With the description of the trivial subbundles of the normal bundles of lines in X as above, we get the following corollaries.

Corollary 20. *With the setup as in Notation 15, we have the following statements.*

(1) *The lines L_1 and L_n are typical in X.*
(2) *The direction T_qL_1 lies outside the trivial subbundle of $\mathcal{N}_{L_n|X}$.*
(3) *The direction T_qL_n lies outside in the trivial subbundle of $\mathcal{N}_{L_1|X}$.*

Corollary 21. *With the setup as in Notation 15, we have the following statements.*

(1) *The lines $L_2,\dots,L_{n-2}$ are typical in X.*
(2) *The direction T_pL_1 lies outside the trivial subbundle of $\mathcal{N}_{L_i|X}$ for $2 \le i \le n-1$.*
(3) *The directions $T_pL_2,\dots,T_pL_{n-1}$ span the normal bundle $\mathcal{N}_{L_1|X}$ at p.*

4. Proof of the Main Theorem

Lemma 22. *Let C be the union of n lines $L_1,\dots,L_n$ in $\mathbb{P}^n$ as in Notation 15. The following properties hold for C for every positive integer d:*

(1) $h^0(C,\mathcal{O}_C(d)) = nd+1$ *and* $h^1(C,\mathcal{O}_C(d)) = 0$.
(2) $h^1(C,\mathcal{I}_C(d)) = 0$.
(3) $h^0(C,\mathcal{I}_C(d)) = h^0(\mathbb{P}^n,\mathcal{O}(d)) - nd - 1$.

Remark 23. The curve C is an example of curves with rational n-fold point, cf. [1, 3.7]. The following lemma is an analogue of [1, Lemma 3.8].

Proof. This can be computed directly. For any $d > 0$, when $i = 1,\dots,n-1$, the homogeneous polynomials of degree d that do not vanish on L_i are generated by $\{x_0^d, x_0^{d-1}x_i,\dots,x_i^d\}$. The homogeneous polynomials of degree d that do not vanish on L_n are generated by $\{x_1^d, x_1^{d-1}x_i,\dots,x_n^d\}$. Since every global section of $\mathcal{O}_C(d)$ is obtained by gluing global sections on each component, which imposes exactly $n-1$ linear conditions, we have

$$h^0(\mathcal{O}_C(d)) = n(d+1)-(n-1) = nd+1$$

and $h^0(\mathscr{O}_{\mathbb{P}^n}(d)) \to h^0(\mathscr{O}_C(d))$ is surjective for any d. In particular, the arithmetic genus of C is zero. Condition (1) is proved. The rest of the lemma follows by considering the long exact sequence in cohomology

$$0 \longrightarrow h^0(C, \mathscr{I}_C(d)) \longrightarrow h^0(\mathscr{O}_{\mathbb{P}^n}(d)) \longrightarrow h^0(\mathscr{O}_C(d)) \longrightarrow h^1(C, \mathscr{I}_C(d)).$$ □

Construction 24. *Let C be the union of n lines $L_1, \dots, L_n$ in $\mathbb{P}^n$ as in Notation 15. If we consider $L_1 \cup L_n$ as a conic in $\mathbb{P}^n$, there exists a smooth affine pointed curve $(T, 0)$ and a smoothing $D' \to (T, 0)$ satisfying the following conditions:*

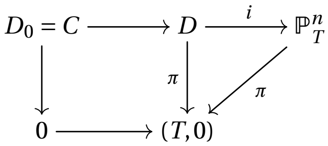
1. *The special fiber D'_0 is $L_1 \cup L_n$;*
2. *For any $t \in T - \{0\}$, D'_t is a smooth conic contained in the plane spanned by L_1 and L_n.*

We may assume that there exists $n-2$ sections $s_i : (T, 0) \to D'$ for $i = 1, \dots, n-2$ such that $s_i(0) = p$ for all i's and for $t \in T - \{0\}$, $s_i(t)$'s are all distinct on D'_t.

For any $s_i(t)$, there exists a unique line $L_{i+1}(t)$ through $s_i(t)$ parallel to L_{i+1}. After gluing the families of lines $L_{i+1}(t)$ on D'_t at $s_i(t)$ for all i's, we get a family of reducible curves $\pi : D \to (T, 0)$ satisfying the following conditions:

1. *The special fiber D_0 is C constructed as in Notation 15.*
2. *For any $t \in T - \{0\}$, D_t is a comb with the handle D'_t and with the teeth lines.*

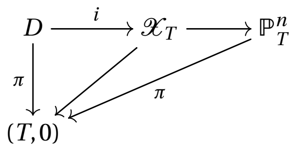
We have the following diagram.



Lemma 25. *The family $\pi : D \to (T, 0)$ is flat. Furthermore, $\pi_* \mathscr{I}_D(d)$ is locally free on T for any $d > 0$, where $\mathscr{I}_D$ is the ideal sheaf of D in $\mathbb{P}^n_T$.*

Proof. The same computational argument as in the proof of Lemma 22 proves that $h^0(\mathbb{P}^n_t, I_{D_t}(d))$ and $h^1(\mathbb{P}^n_t, I_{D_t}(d))$ are constant for any $t \in T - \{0\}$. Thus the Hilbert polynomial is constant. Hence the family is flat over T. The remaining part of the lemma follows from the cohomology and base change theorem [3, III.12.9]. □

Proof of Theorem 3. The theorem is known for $n = 2, 3$. We can assume that $n \geq 4$. By [5, IV.3.11], it suffices to produce one very free curve on a hypersurface of degree n. By Lemma 25, after shrinking T, hypersurfaces of degree n containing D_t in $\mathbb{P}^n_t$ form a trivial projective bundle over $(T, 0)$. Thus the family $\pi : D \to (T, 0)$ admits a lifting to a flat family of pairs $\pi : (\mathscr{X}_T, D) \to (T, 0)$ in $R_{n,n}$ such that the special fiber $(\mathscr{X}_0, D_0)$ is (X, C) which is constructed in Section 3.



All the following steps of the proof requires to shrink T if necessary. By Proposition 12 and Corollary 20, we may assume that the handle D'_t is a typical conic in $\mathscr{X}_t$ for every $t \in T - \{0\}$. By Proposition 11 and Corollary 21 (1), all the teeth of the comb D_t are typical. Thus for every $t \in T - \{0\}$, we get a typical comb D_t as in Definition 13. Now the theorem follows if we verify the two conditions in Proposition 9. Since they are open conditions, it suffices to check on the special fiber (X, C), which is proved in Corollary 21. □

Proof of Theorem 4. For a general Fano hypersurfaces of degree d in $\mathbb{P}^n$, when $d = n$, this is proved in Theorem 3. When $d < n$, we may choose a general Fano hypersurface Y of degree d in $\mathbb{P}^d$ admitting a very free curve $f : \mathbb{P}^1 \to Y$. Construct the cone X of Y in $\mathbb{P}^n$. Note that Y is the intersection of a projective subspace L of dimension d and X. By the normal bundle exact sequence,

$$0 \longrightarrow TY \longrightarrow TX \longrightarrow \mathscr{N}_{Y|X} \longrightarrow 0$$

the sheaf f^*TY is positive and the sheaf $\mathscr{N}_{Y|X}$ is isomorphic to $\mathscr{N}_{L|\mathbb{P}^n}$, which is positive too. Therefore the pullback bundle f^*TX is positive. □

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