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$\mathbb{A}^1$-equivalence of zero cycles on surfaces, II

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Using recent developments in the theory of mixed motives, we prove that the log Bloch conjecture holds for an open smooth complex surface if the Bloch conjecture holds for its compactification. This verifies the log Bloch conjecture for all $\mathbb{Q}$-homology planes and for open smooth surfaces which are not of log general type.

1. Introduction

Throughout this paper, we work with varieties over the complex numbers.

1A. ***Statement of the main theorem.*** Let U be a smooth quasiprojective algebraic variety. Let

$$a : h_0(U)^0 \to \mathrm{Alb}(U)$$

be the Albanese morphism from the zeroth Suslin homology of degree zero to the Albanese variety of U, and let $T(U) := \ker(a)$ be the Albanese kernel. When U is projective, $h_0(U)$ reduces to the Chow group of zero cycles $\mathrm{CH}_0(U)$. Indeed, we get the classical Albanese map.

In dimension one, the Albanese morphism is well-understood by the classical work of Abel and Jacobi in the projective case, and by Rosenlicht in the open case.

Theorem 1.1 (Abel–Jacobi; Rosenlicht [1952; 1954]). *When* $\dim U = 1$, *the Albanese morphism is an isomorphism.*

The higher-dimensional analogue of Theorem 1.1 is much more subtle, although the torsion part of the Albanese morphism is known.

Theorem 1.2 [Rojtman 1980; Spieß and Szamuely 2003]. *In arbitrary dimension, the Albanese morphism induces an isomorphism on torsion subgroups.*

In this paper, we study the two-dimensional case. In one direction, the log Mumford theorem says that the Albanese morphism fails to be injective as long as $p_g(U) \neq 0$.

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Theorem 1.3 [Mumford 1968; Zhu 2018]. *Let U be a smooth algebraic surface with $p_g(U) \neq 0$. Then $T(U)$ is infinite-dimensional.*

In the other direction, we expect the following conjecture. When U is projective, it is famously known as the Bloch conjecture [1980].

Conjecture 1.4 (log Bloch conjecture). Let U be a smooth algebraic surface with $p_g(U) = 0$. Then

$$T(U) = 0.$$

Using recent developments in the theory of mixed motives [Ayoub and Barbieri-Viale 2009; Ayoub 2011; Barbieri-Viale and Kahn 2016; Ayoub 2017], we prove the following theorem.

Theorem 1.5. *Let (X, D) be a log smooth projective surface pair with interior U. If $p_g(U) = 0$, in particular, $p_g(X) = 0$ as well, then the log Bloch conjecture holds for U if and only if it holds for X.*

Since the Bloch conjecture holds for any smooth projective surface X with $\kappa(X) \leq 1$ [Bloch et al. 1976], our main theorem yields the following corollary.

Corollary 1.6. *The log Bloch conjecture holds for U if $\kappa(X) \leq 1$.* □

Since $\kappa(X) \leq \kappa(U)$, Corollary 1.6 generalizes the result of Bloch, Kas, and Lieberman [Bloch et al. 1976] to open surfaces of $\kappa(U) \leq 1$. It also covers the second author's previous result [Zhu 2018] on the log Bloch conjecture for $\kappa(U) = -\infty$.

Further, we may apply Theorem 1.5 to the case where X is of general type and the Bloch conjecture is true. The Bloch conjecture holds in a great number of cases; see [Bauer et al. 2011; Pedrini and Weibel 2016; Voisin 2014] for recent developments.

During the preparation of this paper, Binda and Krishna [2018] proved more general results in the context of Chow groups with modulus using cycle-theoretic methods.

1B. ***Applications of Theorem 1.5 and Corollary 1.6.*** The birational geometry of open surfaces is developed by Kawamata [1979], while it is almost impossible to hope for a complete classification even for $\kappa(U) \leq 1$. We would like to focus on three special classes of surfaces whose geometry is extremely complicated.

Example 1.7 ($\kappa(U) = -\infty$: **log del Pezzo surfaces**). Let U be the smooth locus of a singular del Pezzo surface of Picard number one with at worst quotient singularities. In general, such singular del Pezzos form an unbounded family. Partial classifications are obtained in [Keel and McKernan 1999] with more than sixty exceptional collections. A difficult theorem of [Keel and McKernan 1999] states that U is log rationally connected. In particular, it implies the log Bloch conjecture for U [Zhu 2018, Proposition 4.3].

Since Theorem 1.5 and Corollary 1.6 do not depend on Keel and McKernan's result, we give a new proof of the following result.

Corollary 1.8. *With the notation as above, we have* $h_0(U) = \mathbb{Z}$.

Example 1.9 ($\kappa(U) = 0$: **log Enriques surfaces**). A projective normal surface Y is said to be a *log Enriques surface* if

(1) Y has at worst quotient singularities;

(2) $NK_Y \sim \mathcal{O}_Y$ for some positive integer N;

(3) $\dim H^1(Y, \mathcal{O}_Y) = 0$.

Since K_Y is $\mathbb{Q}$-Cartier, we define the *index* I of Y to be the smallest positive integer such that $IK_Y \sim \mathcal{O}_Y$. By [Kawamata 1979; Tsunoda 1983; Zhang 1991], the index is bounded by 66, while classically (when Y is smooth projective) it is bounded by 6.

Corollary 1.10. *Let* U *be the smooth locus of a log Enriques surface of index* ≥ 2 *defined as above. Then* $h_0(U) = \mathbb{Z}$.

Log Enriques surfaces are partially classified in [Zhang 1991; 1993; Kudryavtsev 2002; 2004]. There are more than 1000 examples of log Enriques surfaces with δ-invariant 2 [Kudryavtsev 2002].

Proof of Corollaries 1.8 and 1.10. Let (X, D) be a minimal log resolution of U. By Corollary 1.6, the log Bloch conjecture holds in both cases. It suffices to show $q(U) = 0$. Since D is the exceptional set of the resolution of quotient singularities, we have $q(U) = q(X)$. Now the del Pezzo case follows from [Zhang 1989, Lemma 1.1(3)] and the Enriques case from [Zhang 1991, Lemma 1.2]. □

Example 1.11. **$\mathbb{Q}$-homology planes** A smooth surface U is a $\mathbb{Q}$-*homology plane* if $H^i(U, \mathbb{Q}) = H^i(\mathbb{A}^2, \mathbb{Q})$ for any i. A $\mathbb{Q}$-homology plane can have log Kodaira dimension $-\infty$, 0, 1, or 2. Ramanujam [1971] constructed the first homology plane of log general type which is topologically contractible. They are classified for log Kodaira dimension ≤ 1, but there is no thorough classification of $\mathbb{Q}$-homology planes of log general type [Miyanishi 2001, Section 3.4].

As all $\mathbb{Q}$-homology planes are rational [Gurjar and Pradeep 1999], Corollary 1.6 implies the following:

Corollary 1.12. *Let* U *be a* $\mathbb{Q}$-*homology plane. Then the log Bloch conjecture holds, that is,* $h_0(U) = \mathbb{Z}$. □

The Bloch conjecture for fake projective planes remains unknown.

1C. ***Ideas of proof.*** The proof of our main theorem has two major ingredients. One is the work in [Ayoub and Barbieri-Viale 2009; Ayoub 2011; Barbieri-Viale and Kahn 2016] on the derived category of 1-motives, especially the construction of a derived Albanese functor. The use is twofold: first, it gives a motivic interpretation of the Albanese morphism, allowing us to apply results from the theory of mixed motives. Second, it provides a way to eliminate "easy" pieces of the motive of U (essentially 1-motives) while keeping track of the homological realization.

The other ingredient is the famous conservativity conjecture; see [Ayoub 2017]. Regarded as a key conjecture in the study of motives, it notably says that a geometric motive is trivial if and only if its homological realization is trivial. By truncating the motive of U using the derived Albanese functor, we eventually arrive at a motive which has trivial homological realization and whose motivic homology controls the Albanese kernel $T(U)$. Therefore, the conservativity conjecture implies the log Bloch conjecture for U. Part of our main theorem then follows from a special case of the conservativity conjecture proven by Wildeshaus [2015].

1D. ***Notation.*** A *log pair* (X, D) means a variety X with a reduced Weil divisor D. We say that (X, D) is *log smooth* if X is smooth and D is a simple normal crossing divisor on X. A log pair is projective if the ambient variety is projective.

Given any smooth quasiprojective variety U, by the resolution of singularities, we may choose a log smooth projective compactification (X, D) with interior U. We use $\kappa(X, D)$ to denote the *log Kodaira dimension*. We define the *log geometric genus* $p_g(X, D) := \dim H^0(\Omega_X^{\dim X}(\log D))$ and the *log irregularity* $q(X, D) := \dim H^0(\Omega_X^1(\log D))$. Since they do not depend on the compactification, we may write $\kappa(U)$, $p_g(U)$, and $q(U)$ as well.

2. Preliminaries

By Theorem 1.2, it suffices to consider the Albanese morphism with $\mathbb{Q}$-coefficients. From now on, all (co)homology, cycle groups, and motives are taken with $\mathbb{Q}$-coefficients.

2A. ***Mixed motives and conservativity.*** We refer to [Voevodsky et al. 2000; Mazza et al. 2006] for Voevodsky's theory of mixed motives. With $\mathbb{Q}$-coefficients, the categories of mixed motives in the Nisnevich and étale topologies are equivalent.

Let $\mathrm{DM}_{\mathrm{gm}}$ denote the triangulated category of geometric motives, and let $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}$ denote the triangulated category of effective geometric motives. We follow the homological convention. The unit object of $\mathrm{DM}_{\mathrm{gm}}$ is denoted by $\mathbb{Q}(0)$, or simply $\mathbb{Q}$, and the Tate object $\mathbb{Q}(1)$. Given an object $M \in \mathrm{DM}_{\mathrm{gm}}$, its dual object $\mathcal{H}om_{\mathrm{DM}_{\mathrm{gm}}}(M, \mathbb{Q})$ is denoted by $M^\vee$. The motive of a smooth variety Y is denoted by $M(Y) \in \mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}$.

The i-th motivic homology of $M \in \mathrm{DM}_{\mathrm{gm}}$ is defined to be

$$h_i(M) = \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}}(\mathbb{Q}[i], M).$$

For $M = M(Y)$, this recovers the i-th Suslin homology $h_i(Y) = h_i(M(Y))$.

Further, we refer to [Huber 2000] for the Hodge realization functor

$$R^H : \mathrm{DM}_{\mathrm{gm}} \to D^b(\mathrm{MHS}).$$

Here we use the covariant version of R^H. Composing with the forgetful functor $D^b(\mathrm{MHS}) \to D^b(\mathbb{Q})$, we obtain the Betti realization

$$R^B : \mathrm{DM}_{\mathrm{gm}} \to D^b(\mathbb{Q}).$$

Recall the statement of the conservativity conjecture.

Conjecture 2.1 (see [Ayoub 2017, Conjecture 2.1]). The Betti realization functor R^B is conservative. In other words, a morphism $f : M \to N$ in $\mathrm{DM}_{\mathrm{gm}}$ is an isomorphism if and only if $R^B(f) : R^B(M) \to R^B(N)$ is an isomorphism.

Using consequences of the standard conjecture D for abelian varieties [André and Kahn 2002], Kimura–O'Sullivan finiteness [Kimura 2005], and Bondarko's weight structures [2009; 2010], Wildeshaus proved the following special case of the conservativity conjecture.

Theorem 2.2 [Wildeshaus 2015, Theorem 1.12]. *Let* $\mathrm{DM}^{\mathrm{ab}}_{\mathrm{gm}} \subset \mathrm{DM}_{\mathrm{gm}}$ *denote the smallest triangulated subcategory containing the motives of smooth curves and closed under direct summands, tensor products, and duality. Then the restriction of* R^B *to* $\mathrm{DM}^{\mathrm{ab}}_{\mathrm{gm}}$ *is conservative.*

With the notion of $\mathrm{DM}^{\mathrm{ab}}_{\mathrm{gm}}$, we now state our main theorem extending Theorem 1.5.

Theorem 2.3. *Let* (X, D) *be a log smooth projective surface pair with interior* U*. Then the following four conditions are equivalent*:

(1) $T(U) = 0$;

(2) $T(X) = 0$;

(3) $M(U) \in \mathrm{DM}^{\mathrm{ab}}_{\mathrm{gm}}$;

(4) $M(X) \in \mathrm{DM}^{\mathrm{ab}}_{\mathrm{gm}}$.

2B. ***Derived category of* 1*-motives.*** We mainly follow the book [Barbieri-Viale and Kahn 2016]. Let $\mathcal{M}_1$ denote Deligne's category of 1-motives [Deligne 1974] with $\mathbb{Q}$-coefficients. By [Orgogozo 2004, Théorème 3.4.1], the bounded derived category $D^b(\mathcal{M}_1)$ can be naturally identified with the thick triangulated subcategory of $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ generated by the motives of smooth curves, denoted by $d_{\leq 1}\,\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$. The identification is compatible with realizations [Vologodsky 2012]. For simplicity we always make this identification.

One of the main results of [Barbieri-Viale and Kahn 2016] is the construction of a *derived Albanese functor*.

Theorem 2.4 [Barbieri-Viale and Kahn 2016, Corollary 6.2.2]. *The inclusion*

$$d_{\leq 1}\,\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}} \hookrightarrow \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$$

admits a left adjoint

$$L\,\mathrm{Alb} : \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}} \to d_{\leq 1}\,\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}\,.$$

We list a number of results and facts about the functor $L\,\mathrm{Alb}$ which will be used in the proof of our main theorem. To begin with, when Y is a smooth variety, we write $L\,\mathrm{Alb}(Y) = L\,\mathrm{Alb}(M(Y))$. Then the natural morphism $M(Y) \to L\,\mathrm{Alb}(Y)$ induces a morphism in motivic homology

$$h_0(Y) \to h_0(L\,\mathrm{Alb}(Y)). \tag{2.5}$$

By [Barbieri-Viale and Kahn 2016, Lemma 13.4.2], we have

$$h_0(L\,\mathrm{Alb}(Y))^0 = \mathrm{Alb}(Y) \otimes \mathbb{Q},$$

and the degree zero part of (2.5) coincides with the Albanese morphism.

The next statement concerns the Hodge realization of $L\,\mathrm{Alb}(M)$ for $M \in \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$. Recall that a mixed Hodge structure H is *effective* if the (i, j)-th part of the weight-graded piece $\mathrm{Gr}^W_{i+j} H$ vanishes unless $i, j \leq 0$. Given an effective mixed Hodge structure H, let $H_{\leq 1}$ denote the maximal quotient of H of weights ≥ -2 and of types $(0, 0)$, $(0, -1)$, $(-1, 0)$, and $(-1, -1)$.

Theorem 2.6 [Barbieri-Viale and Kahn 2016, Theorem 15.3.1]. *For $M \in \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$, the morphism $M \to L\,\mathrm{Alb}(M)$ induces isomorphisms*

$$H_i(R^H(M))_{\leq 1} \xrightarrow{\sim} H_i\big(R^H(L\,\mathrm{Alb}(M))\big).$$

The theorem above applies to $L\,\mathrm{Alb}(Y)$ and also to the Borel–Moore variant of $L\,\mathrm{Alb}(Y)$. Let $M^c(Y) \in \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ denote the motive of Y with compact support. Note that by [Voevodsky et al. 2000, Chapter 5, Theorem 4.3.7], there is an isomorphism

$$M^c(Y) \simeq M(Y)^\vee(\dim Y)[2\dim Y]. \tag{2.7}$$

We write $L\,\mathrm{Alb}^c(Y) = L\,\mathrm{Alb}(M^c(Y))$.

Corollary 2.8 [Barbieri-Viale and Kahn 2016, Corollary 15.3.2]. *By Theorem 2.6, we have*

$$H_i\big(R^H(L\,\mathrm{Alb}(Y))\big)=\begin{cases}H_0(Y,\mathbb{Q}), & i=0,\\ H_1(Y,\mathbb{Q}), & i=1,\\ H_2(Y,\mathbb{Q})_{\le 1}, & i=2,\\ 0, & i<0 \textit{ or } i>2,\end{cases}$$

$$H_i\big(R^H(L\,\mathrm{Alb}^c(Y))\big)=\begin{cases}H_0^{\mathrm{BM}}(Y,\mathbb{Q}), & i=0,\\ H_1^{\mathrm{BM}}(Y,\mathbb{Q}), & i=1,\\ H_i^{\mathrm{BM}}(Y,\mathbb{Q})_{\le 1}, & 2\le i\le \dim Y+1,\\ 0, & i<0 \textit{ or } i>\dim Y+1.\end{cases}$$

Finally, we recall the fact that $\mathcal{M}_1$ is of cohomological dimension one [Orgogozo 2004, Proposition 3.2.4]. Hence, all elements in $D^b(\mathcal{M}_1)$ can be represented by complexes with zero differentials. In particular, we have

$$L\,\mathrm{Alb}(Y)\simeq\bigoplus_{i=0}^{2}L_i\,\mathrm{Alb}(Y)[i]\quad\text{and}\quad L\,\mathrm{Alb}^c(Y)\simeq\bigoplus_{i=0}^{\dim Y+1}L_i\,\mathrm{Alb}^c(Y)[i],\tag{2.9}$$

with $L_i\,\mathrm{Alb}(Y)$, $L_i\,\mathrm{Alb}^c(Y)\in\mathcal{M}_1$; see [Barbieri-Viale and Kahn 2016, Corollary 9.2.3 and Proposition 10.6.2]. When $\dim Y=1$, this gives the "Chow–Künneth" decomposition of $M(Y)$ [Barbieri-Viale and Kahn 2016, Corollary 11.1.1]

$$M(Y)\simeq L\,\mathrm{Alb}(Y)\simeq\bigoplus_{i=0}^{2}L_i\,\mathrm{Alb}(Y)[i].\tag{2.10}$$

3. Proof of the main theorem

In this section we prove our main theorem, that is, Theorem 2.3.

3A. ***Proof of (1) ⇒ (2) ⇒ (3) ⇔ (4).*** For (1) ⇒ (2), consider a partial compactification $U\subset Y\subset X$ such that $C=Y\setminus U$ is a smooth curve. By induction, it suffices to show that $T(U)=0$ implies $T(Y)=0$.

Recall the Gysin distinguished triangle [Voevodsky et al. 2000, Chapter 5, Proposition 3.5.4]

$$M(U)\to M(Y)\to M(C)(1)[2]\to M(U)[1].\tag{3.1}$$

By applying the functor $L\,\mathrm{Alb}$, we find a morphism of distinguished triangles

$$\begin{tikzcd} M(U) \arrow[r] \arrow[d] & M(Y) \arrow[r] \arrow[d] & M(C)(1)[2] \arrow[r] \arrow[d] & M(U)[1] \arrow[d] \\ L\,\mathrm{Alb}(U) \arrow[r] & L\,\mathrm{Alb}(Y) \arrow[r] & \mathbb{Q}(1)[2] \arrow[r] & L\,\mathrm{Alb}(U)[1] \end{tikzcd}\tag{3.2}$$

Here we used the fact that $L\,\mathrm{Alb}(M(C)(1))\simeq\mathbb{Q}(1)$ [Barbieri-Viale and Kahn 2016, Proposition 8.2.3]. Moreover, the morphism

$$M(C)(1) \to L\,\mathrm{Alb}(M(C)(1)) \simeq \mathbb{Q}(1)$$

coincides with the projection in (2.10),

$$M(C) \to L_0\,\mathrm{Alb}(C) \simeq \mathbb{Q},$$

twisted by $\mathbb{Q}(1)$.

Now we apply motivic homology to the distinguished triangles in (3.2). Since $h_0(U) \to h_0(Y)$ is surjective [Zhu 2018, Lemma 4.2] and

$$h_0(\mathbb{Q}(1)[2]) = \mathrm{CH}_{-1}(\mathrm{pt}) = 0,$$

we obtain a commutative diagram with exact rows

$$\begin{tikzcd}
h_1(M(C)(1)[2]) \arrow[r] \arrow[d] & h_0(U) \arrow[r] \arrow[d] & h_0(Y) \arrow[r] \arrow[d] & 0 \arrow[d] \\
h_1(\mathbb{Q}(1)[2]) \arrow[r] & h_0(L\,\mathrm{Alb}(U)) \arrow[r] & h_0(L\,\mathrm{Alb}(Y)) \arrow[r] & 0
\end{tikzcd}$$

The first vertical arrow is surjective since it comes from a projection. The middle vertical arrows are given by the Albanese morphisms of U and Y. Our assumption $T(U) = 0$ says that the second vertical arrow is injective. Then, by the five lemma, the third vertical arrow is also injective, and hence $T(Y) = 0$.

The implication (2) $\Rightarrow$ (4) is essentially due to Guletskiĭ and Pedrini [2003, Theorem 7]. The precise statement can be found in [Kahn et al. 2007, Corollary. 4.9], where it is shown that $T(X) = 0$ is equivalent to the vanishing of the transcendental part in the Chow–Künneth decomposition of $M(X)$, and that the remaining parts belong to $\mathrm{DM}^{\mathrm{ab}}_{\mathrm{gm}}$. Further, by the distinguished triangle (3.1) and the fact that $M(C) \in \mathrm{DM}^{\mathrm{ab}}_{\mathrm{gm}}$, we see that $M(U) \in \mathrm{DM}^{\mathrm{ab}}_{\mathrm{gm}}$ if and only if $M(Y) \in \mathrm{DM}^{\mathrm{ab}}_{\mathrm{gm}}$. The equivalence (3) $\Leftrightarrow$ (4) then follows by induction. $\square$

3B. ***Proof of (3) ⇒ (1).*** We define the motive $M'(U) \in \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ by the distinguished triangle

$$M'(U) \to M(U) \to L\,\mathrm{Alb}(U) \to M'(U)[1]. \tag{3.3}$$

Our assumption $p_g(U) = 0$ says that $H_2(U, \mathbb{Q}) = H_2(U, \mathbb{Q})_{\leq 1}$. Then, by Theorem 2.6 and Corollary 2.8, we have

$$H_i\big(R^B(M'(U))\big) = \begin{cases} H_3(U, \mathbb{Q}), & i = 3, \\ H_4(U, \mathbb{Q}), & i = 4, \\ 0, & i < 3 \text{ or } i > 4. \end{cases}$$

Now consider the motive $M'(U)^\vee(2)[4]$. By the duality

$$H_i^{\mathrm{BM}}(U, \mathbb{Q}) = H_{4-i}(U, \mathbb{Q})^\vee(2),$$

we have

$$H_i\big(R^B(M'(U)^\vee(2)[4])\big) = \begin{cases} H_0^{\mathrm{BM}}(U, \mathbb{Q}), & i = 0, \\ H_1^{\mathrm{BM}}(U, \mathbb{Q}), & i = 1, \\ 0, & i < 0 \text{ or } i > 1. \end{cases} \tag{3.4}$$

There is a dual distinguished triangle to (3.3),

$$L\,\mathrm{Alb}(U)^\vee(2)[4] \to M(U)^\vee(2)[4] \to M'(U)^\vee(2)[4] \to L\,\mathrm{Alb}(U)^\vee(2)[5].$$

By (2.7), we have $M(U)^\vee(2)[4] \in \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$. Since $L\,\mathrm{Alb}(U)^\vee(2)[4] \in \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ by Cartier duality [Barbieri-Viale and Kahn 2016, Proposition 4.5.1], we also have $M'(U)^\vee(2)[4] \in \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$. This allows us to apply the functor $L\,\mathrm{Alb}$ to $M'(U)^\vee(2)[4]$. By Theorem 2.6 and Corollary 2.8, the morphism

$$M'(U)^\vee(2)[4] \to L\,\mathrm{Alb}(M'(U)^\vee(2)[4]) \tag{3.5}$$

induces an isomorphism

$$R^B(M'(U)^\vee(2)[4]) \xrightarrow{\sim} R^B\big(L\,\mathrm{Alb}(M'(U)^\vee(2)[4])\big).$$

We are ready to apply conservativity. Our assumption $M(U) \in \mathrm{DM}^{\mathrm{ab}}_{\mathrm{gm}}$ implies $M(U)^\vee(2)[4] \in \mathrm{DM}^{\mathrm{ab}}_{\mathrm{gm}}$. Furthermore, since $d_{\le 1}\,\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}} \subset \mathrm{DM}^{\mathrm{ab}}_{\mathrm{gm}}$, we have $L\,\mathrm{Alb}(U)^\vee(2)[4] \in \mathrm{DM}^{\mathrm{ab}}_{\mathrm{gm}}$ and thus $M'(U)^\vee(2)[4] \in \mathrm{DM}^{\mathrm{ab}}_{\mathrm{gm}}$. Then, according to Theorem 2.2, the morphism (3.5) is itself an isomorphism.

We thus obtain from (3.3) a distinguished triangle

$$\begin{aligned} L\,\mathrm{Alb}(M'(U)^\vee(2)[4])^\vee(2)[4] &\to M(U) \to L\,\mathrm{Alb}(U) \\ &\to L\,\mathrm{Alb}(M'(U)^\vee(2)[4])^\vee(2)[5]. \end{aligned} \tag{3.6}$$

Taking motivic homology, we have an exact sequence

$$h_0\big(L\,\mathrm{Alb}(M'(U)^\vee(2)[4])^\vee(2)[4]\big) \to h_0(U) \to h_0(L\,\mathrm{Alb}(U)),$$

where the second arrow is given by the Albanese morphism of U. Hence, to prove $T(U) = 0$, it suffices to show that

$$h_0\big(L\,\mathrm{Alb}(M'(U)^\vee(2)[4])^\vee(2)[4]\big) = 0.$$

By [Deligne 1974, Construction 10.1.3], the Hodge realization gives a full embedding $\mathcal{M}_1 \subset \mathrm{MHS}$. A comparison of realizations yields the isomorphisms

$$\begin{aligned}L\,\mathrm{Alb}(M'(U)^\vee(2)[4]) &\simeq \bigoplus_{i=0}^{1} L_i\,\mathrm{Alb}(M'(U)^\vee(2)[4])[i]\\ &\simeq \bigoplus_{i=0}^{1} L_i\,\mathrm{Alb}(M(U)^\vee(2)[4])[i]\\ &\simeq \bigoplus_{i=0}^{1} L_i\,\mathrm{Alb}^c(U)[i]. \end{aligned}\tag{3.7}$$

More precisely, the first isomorphism is a consequence of (2.9) and (3.4). The last two isomorphisms follow from the duality (2.7), Corollary 2.8, and (3.4). Alternatively, one may also deduce (3.7) from Theorem 2.2 since all motives involved belong to $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{ab}}$.

We compute

$$\begin{aligned}&h_0\big(L\,\mathrm{Alb}(M'(U)^\vee(2)[4])^\vee(2)[4]\big)\\ &\qquad= h_0\bigg(\bigoplus_{i=0}^{1}(L_i\,\mathrm{Alb}^c(U)[i])^\vee(2)[4]\bigg)\\ &\qquad= \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}}\bigg(\mathbb{Q}, \bigoplus_{i=0}^{1}(L_i\,\mathrm{Alb}^c(U)[i])^\vee(2)[4]\bigg)\\ &\qquad= \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}}\bigg(\bigoplus_{i=0}^{1} L_i\,\mathrm{Alb}^c(U)[i], \mathbb{Q}(2)[4]\bigg)\\ &\qquad= \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}}(L_0\,\mathrm{Alb}^c(U), \mathbb{Q}(2)[4]) \oplus \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}}(L_1\,\mathrm{Alb}^c(U), \mathbb{Q}(2)[3]).\end{aligned}$$

By [Barbieri-Viale and Kahn 2016, Proposition 10.6.2], we have

$$L_0\,\mathrm{Alb}^c(U) \simeq \begin{cases}\mathbb{Q} & \text{if } U \text{ is projective,}\\ 0 & \text{if not.}\end{cases}$$

Since

$$\mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}}(\mathbb{Q}, \mathbb{Q}(2)[4]) = \mathrm{CH}_{-2}(\mathrm{pt}) = 0,$$

we find in both cases $\mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}}(L_0\,\mathrm{Alb}^c(U), \mathbb{Q}(2)[4]) = 0$.

Further, by [Barbieri-Viale and Kahn 2016, Corollary 12.11.2], the 1-motive $L_1\,\mathrm{Alb}^c(U)$ is represented by a two-term complex in degrees 0 and 1

$$\mathbb{Q}^{\oplus r} \to A \otimes \mathbb{Q},$$

where A is an abelian variety and $r = \#\{\text{connected components of } D\} - 1$. In other words, there is an extension of 1-motives

$$0 \to (A \otimes \mathbb{Q})[-1] \to L_1\,\mathrm{Alb}^c(U) \to \mathbb{Q}^{\oplus r} \to 0, \tag{3.8}$$

which yields an exact sequence

$$\mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}}(\mathbb{Q}, \mathbb{Q}(2)[3])^{\oplus r} \to \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}}(L_1 \operatorname{Alb}^c(U), \mathbb{Q}(2)[3]) \\ \to \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}}((A \otimes \mathbb{Q})[-1], \mathbb{Q}(2)[3]).$$

Since

$$\mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}}(\mathbb{Q}, \mathbb{Q}(2)[3]) = \mathrm{CH}_{-2}(\mathrm{pt}, 1) = 0,$$

it suffices to show that $\mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}}((A \otimes \mathbb{Q})[-1], \mathbb{Q}(2)[3]) = 0$.

We may assume A to be the Albanese variety of a smooth projective surface S. If $\dim A > 2$, the surface S is obtained by taking a sequence of general hyperplane sections of A. Then we have $\mathrm{Alb}(S) \simeq \mathrm{Alb}(A) \simeq A$ by the Lefschetz hyperplane theorem. Recall the Chow–Künneth decomposition of $M(S)$ [Murre 1990, Theorem 3]:

$$M(S) \simeq \bigoplus_{i=0}^{4} M_i(S)[i].$$

We have $M_{4-i}(S) \simeq M_i(S)^\vee(2)$ and $M_1(S) \simeq (A \otimes \mathbb{Q})[-1]$. Hence

$$\begin{aligned} \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}}((A \otimes \mathbb{Q})[-1], \mathbb{Q}(2)[3]) &= \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}}(M_1(S), \mathbb{Q}(2)[3]) \\ &= \mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}}(\mathbb{Q}, M_3(S)[3]) \\ &= \mathrm{CH}_0(M_3(S)[3]) \\ &= 0, \end{aligned} \tag{3.9}$$

where the last equality follows again from [Murre 1990, Theorem 3]. The proof of Theorem 2.3 is now complete. □

3C. *"Chow–Künneth" decomposition.* Our proof of Theorem 2.3 also leads to the following consequence.

Corollary 3.10. *Assume one of the equivalent conditions in Theorem 2.3. Then $M(U)$ admits a "Chow–Künneth" decomposition*

$$M(U) \simeq \bigoplus_{i=0}^{2} L_i \operatorname{Alb}(U)[i] \oplus \bigoplus_{i=3}^{4} L_{4-i} \operatorname{Alb}^c(U)^\vee(2)[i].$$

In particular, it is Kimura–O'Sullivan finite.

Proof. Consider the distinguished triangle (3.6) obtained under the assumption $M(U) \in \mathrm{DM}^{\mathrm{ab}}_{\mathrm{gm}}$. By (2.9) and (3.7), there are isomorphisms

$$L \operatorname{Alb}(U) \simeq \bigoplus_{i=0}^{2} L_i \operatorname{Alb}(U)[i] \quad \text{and} \quad L \operatorname{Alb}(M'(U)^\vee(2)[4]) \simeq \bigoplus_{i=0}^{1} L_i \operatorname{Alb}^c(U)[i].$$

Hence (3.6) induces a distinguished triangle

$$\bigoplus_{i=3}^{4} L_{4-i}\,\mathrm{Alb}^{c}(U)^{\vee}(2)[i] \to M(U) \to \bigoplus_{i=0}^{2} L_{i}\,\mathrm{Alb}(U)[i] \\ \to \bigoplus_{i=3}^{4} L_{4-i}\,\mathrm{Alb}^{c}(U)^{\vee}(2)[i+1].$$

For the distinguished triangle to split, it suffices to show that

$$\mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}}\Biggl(\bigoplus_{i=0}^{2} L_{i}\,\mathrm{Alb}(U)[i], \bigoplus_{i=3}^{4}(L_{4-i}\,\mathrm{Alb}^{c}(U))^{\vee}(2)[i+1]\Biggr) = 0.$$

The left-hand side consists of six direct summands, all of which can be computed explicitly. To keep the paper short we only do the most complicated one, that is,

$$\mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}}(L_1\,\mathrm{Alb}(U)[1], L_1\,\mathrm{Alb}^{c}(U)^{\vee}(2)[4]). \tag{3.11}$$

By [Barbieri-Viale and Kahn 2016, Corollary 9.2.3], the 1-motive $L_1\,\mathrm{Alb}(U)$ is represented by the two-term complex in degrees 0 and 1

$$0 \to \mathrm{Alb}(U)\otimes\mathbb{Q}.$$

Since the abelian part of the semiabelian variety $\mathrm{Alb}(U)$ is $\mathrm{Alb}(X)$, this gives an extension of 1-motives

$$0 \to (\mathbb{G}_m\otimes\mathbb{Q})^{\oplus s}[-1] \to L_1\,\mathrm{Alb}(U) \to (\mathrm{Alb}(X)\otimes\mathbb{Q})[-1] \to 0. \tag{3.12}$$

We have $(\mathbb{G}_m\otimes\mathbb{Q})[-1]\simeq\mathbb{Q}(1)$ and $(\mathrm{Alb}(X)\otimes\mathbb{Q})[-1]\simeq M_1(X)$.

Combining (3.8) and (3.12), we see that (3.11) sits in the middle of several extensions involving the following four terms:

(1) $\mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}}(M_1(X)[1], M_3(S)[4])$;

(2) $\mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}}(M_1(X)[1], \mathbb{Q}(2)[4])$;

(3) $\mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}}(\mathbb{Q}(1)[1], M_3(S)[4])$;

(4) $\mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}}(\mathbb{Q}(1)[1], \mathbb{Q}(2)[4])$.

The vanishing of the second term is shown in (3.9) (with X replaced by S). The vanishing of the three other terms follows from the fact that given two Chow motives M and M', we have $\mathrm{Hom}_{\mathrm{DM}_{\mathrm{gm}}}(M, M'[i]) = 0$ for all $i > 0$. This in turn follows from [Voevodsky et al. 2000, Chapter 5, Corollary 4.2.6] and the cancellation theorem [Voevodsky et al. 2000, Chapter 5, Theorem 4.3.1]. Hence (3.11) vanishes.

Finally, by [Mazza 2004, Remark 5.11], all elements in $d_{\leq 1}\,\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}$ are Kimura–O'Sullivan finite. The last statement follows since Kimura–O'Sullivan finiteness is closed under direct sums and tensor products. □

On the other hand, there exist motives of smooth surfaces which are not Kimura–O'Sullivan finite [Mazza 2004, Theorem 5.18].

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